TRANSFER OF CHARACTERS IN THE THETA CORRESPONDENCE WITH ONE COMPACT MEMBER

ALLAN MERINO

Abstract. Here are the notes of a talk I gave during the "SL2R Workshop" at the University of Nancy, France, in December 2017. For more details, see my paper [11].

Contents

1. Character theory

Let G be a Lie group and let (Π, V) be a finite dimensional representation of G. The function Θ_Π

$$\mathrm{G} \ni g \to \Theta_\Pi(g) = \mathrm{tr}(\Pi(g)) \in \mathbb{C}$$

is well defined: this is the character of the representation (Π, V) of G.
Assume that G is compact and connected. Let T be a torus in G. We denote by $\mathfrak{t}_0$ and $\mathfrak{g}_0$ the Lie algebras of T and G respectively, and by $\mathfrak{t}$ and $\mathfrak{g}$ their complexifications. Let $\Psi^+(\mathfrak{g}, \mathfrak{t})$ be a system of positives roots and $\mathscr{W}(\mathfrak{g}, \mathfrak{t})$ the corresponding Weyl group. Then every irreducible representation Π of G is finite-dimensional and uniquely determined (up to equivalence) by its character Θ_Π. Fix (Π_λ, V_λ) an irreducible representation of highest weight $\lambda \in \mathfrak{t}^*$. Hermann Weyl proved that the character Θ_{Π_λ} is given by:

$$\Theta_{\Pi_\lambda}(\exp(X)) = \sum_{\sigma \in \mathscr{W}(\mathfrak{g},\mathfrak{t})} \varepsilon(\sigma) \frac{e^{\sigma(\lambda+\rho)(X)}}{\prod\limits_{\alpha \in \Psi^+(\mathfrak{g},\mathfrak{t})} (e^{\frac{\alpha(X)}{2}} - e^{-\frac{\alpha(X)}{2}})}, \qquad (X \in i\mathfrak{t}_0) ,$$

where $\rho = \frac{1}{2} \sum\limits_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} \alpha \in \mathfrak{t}^*$.

Problem: If V is infinite dimensional, the map Θ_Π is not necessarily well defined.
Solution: Harish-Chandra extended the concept of character to a certain class of representations of a reductive Lie group: the quasi-simple representations (see [3, Section 10]).
Let G be a real reductive Lie group and $(\Pi, \mathscr{H})$ be a quasi-simple representation. We denote by $\mathscr{C}_c^\infty(\mathrm{G})$ the space of smooth and compactly supported functions on G. For every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, we denote by $\Pi(\Psi)$

the bounded operator on $\mathscr{H}$ given by:

$$\Pi(\Psi) = \int_{G} \Psi(g)\Pi(g)dg\,.$$

Theorem 1.1. *For every* $\Psi \in \mathscr{C}_c^\infty(G)$*,* $\Pi(\Psi)$ *is a trace class operator and the corresponding map:*

$$\Theta_\Pi : \mathscr{C}_c^\infty(G) \ni \Psi \to \mathrm{tr}(\Pi(\Psi)) \in \mathbb{C}$$

is a distribution (in the sense of Laurent Schwartz).
Moreover, there exists a locally integrable function Θ_Π *on* G*, analytic on* G^{reg}*, such that* $\Theta_\Pi = T_{\Theta_\Pi}$*, i.e. for every* $\Psi \in \mathscr{C}_c^\infty(G)$*,*

$$\Theta_\Pi(\Psi) = \int_{G} \Psi(g)\Pi(g)dg\,.$$

Proof. See [4, Section 5] and [5, Theorem 2].

□

Harish-Chandra proved in [6] that G has discrete series representations if and only if G has a compact Cartan subgroup. Assume that G has such a Cartan subgroup T and let K be a maximal compact subgroup of G s.t. $T \subseteq K \subseteq G$. Let $(\Pi, \mathscr{H}_\Pi)$ be a discrete series representation of G with Harish-Chandra parameter λ (see [9, Proposition 9.6] and [9, Theorem 9.20]). We will denote by Θ_{Π_λ} the character of the representation Π_λ and let $\mathfrak{t}_0$ be the Lie algebra of T. For every $X \in \mathfrak{t}_0^{\mathrm{reg}}$, we have:

$$\Theta_{\Pi_\lambda}(\exp(X)) = (-1)^{\frac{1}{2}\dim(G/K)} \sum_{w \in \mathscr{W}(\mathfrak{k})} \varepsilon(w) \frac{e^{w\lambda(X)}}{\prod\limits_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} (e^{\frac{\alpha(X)}{2}} - e^{-\frac{\alpha(X)}{2}})}\,, \qquad (x \in \mathfrak{t}_0)\,,$$

where $\mathscr{W}(\mathfrak{k}) = N_K(T)/T$ is the compact Weyl group (see [6]).
Many other authors had been interested in characters theory. We can for example mentionned Rossmann ([12]), who proved Kirillov formula (see [8]) for discrete series representations, and Enright ([2]), who, using cohomological methods, established a formula for the character of a unitary highest weight module of a simple Lie group. Quickly, let (G, K) be an hermitian symmetric pair with G simple and $(\Pi, \mathscr{H}_\Pi)$ be a representation of G of highest weight $\lambda - \rho$. Then, the character Θ_Π is given by the following formula:

$$\prod_{\alpha>0}(e^{\frac{\alpha(x)}{2}} - e^{-\frac{\alpha(x)}{2}})\Theta_\Pi(\exp(x)) = \sum_{\omega \in \mathscr{W}_\lambda^{\mathfrak{k}}} (-1)^{l_\lambda(\omega)}\Theta(K, \Lambda(\omega,\lambda))(\exp(x))\,, \qquad (x \in \mathfrak{t}^{\mathrm{reg}})\,, \tag{1}$$

where $\mathscr{W}_\lambda^{\mathfrak{k}}$ is defined in [2], definition 2.1, and $\Theta(K, \Lambda(\omega,\lambda))(\exp(x))$ is the character of a K-representation of highest weight $\Lambda(\omega,\lambda)$, with $\Lambda(\omega,\lambda)$ is defined in [2, Corollary 2.3].

2. The Weil (or metaplectic) representation

A classical construction of this representation uses the Stone-Von Neumann theorem (in fact, using the Heisenberg group and their representations, we obtain a projective representation of Sp(W), and hence a representation of a double cover of Sp(W), the metaplectic group). We can for example consult [10]. But here, I recall and use a much more explicit construction of this representation, done by Anne-Marie Aubert and Tomasz Przebinda and published in 2016 (see [1])
Let $(W, \langle\cdot,\cdot\rangle)$ be a real symplectic vector space and let $W = X \oplus Y$ be a complete polarisation of W. Let Sp(W) be the symplectic group associated with $(W, \langle\cdot,\cdot\rangle)$ and J a positive complex structure on W, i.e. $J \in \mathfrak{sp}(W)$, $J^2 = -1$ such that the bilinear symetric form $\langle J\cdot,\cdot\rangle$ is positive definite. For every $g \in G$,

we define $J_g = J^{-1}(g-1)$. It is easy to show that the restriction $J_g : J_g W \to J_g W$ is well defined and invertible.

Metaplectic group:

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \{\widetilde{g} = (g,\xi) \in \mathrm{Sp}(\mathrm{W}) \times \mathbb{C}^*, \xi^2 = \det((-i\mathrm{J}_g)_{|_{\mathrm{J}_g\mathrm{W}}})^{-1}\}\,.$$

Remark: If $\det(g-1) \neq 0$, then $\xi^2 = \det(i(g-1))^{-1}$. Denote by pr the natural projection associated to the covering $\widetilde{\mathrm{Sp}}(\mathrm{W})$ of $\mathrm{Sp}(\mathrm{W})$

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \widetilde{g} = (g,\xi) \to \mathrm{pr}(\widetilde{g}) = g \in \mathrm{Sp}(\mathrm{W})\,,$$

and let χ be the character of $\mathbb{R}$ given by $\chi(r) = e^{2i\pi r}$. We consider

$$\Theta(\widetilde{g}) = \xi\,, \qquad \mathrm{T}(\widetilde{g}) = \Theta(\widetilde{g})t(g) = \Theta(\widetilde{g})\chi_{c(g)}\mu_{(g-1)W}\,, \qquad \left(\widetilde{g} = (g,\xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W})\right),$$

where $c(g), g \in \mathrm{Sp}(\mathrm{W})$, is the Cayley transform of g defined on the space $(g-1)\mathrm{W}$ by

$$c(g) : (g-1)\mathrm{W} \ni (g-1)w \to (g+1)w + \mathrm{Ker}(g-1) \in \mathrm{W}/\mathrm{Ker}(g-1)\,,$$

(see [1, Section 2.3] for more details), $\chi_{c(g)} : \mathrm{W} \to \mathbb{C}$ is given by $\chi_{c(g)}(w) = \chi(\frac{1}{4}\langle c(g)w, w\rangle)$, and $\mu_{(g-1)\mathrm{W}}$ is the Lebesgue measure on $(g-1)\mathrm{W}$ (we fix a normalization for the Lebesgue measure of every subspace of W so that the unit cube with respect to the inner product $\langle\cdot,\cdot\rangle$ is 1).

Twisted convolution: for all $\phi, \eta \in \mathrm{S}(\mathrm{W})$ (the Schwartz space on W), we define the twisted convolution $\phi\natural\eta$ associated to the character χ is given by

$$\phi\natural\eta(w) = \int_{\mathrm{W}} \phi(u)\eta(w-u)\chi(\frac{1}{2}\langle u, w\rangle)d\mu_{\mathrm{W}}(w)\,.$$

We can extend this convolution to some distributions: in particular, for $\phi \in \mathrm{S}(\mathrm{W})$ and $\widetilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, we get:

$$T(\widetilde{g})\natural\phi(w) = \Theta(\widetilde{g}) \int_{\mathrm{W}} \chi_{c(g)}(w)\phi(w-u)\chi(\frac{1}{2}\langle u, w\rangle)\, d\mu_W(w) \in \mathrm{S}(\mathrm{W}).$$

Weyl transform and Schwartz' kernel operator:

$$\mathrm{S}(\mathrm{W}) \ni f \to \mathscr{K}f \in \mathrm{S}(\mathrm{X}\times\mathrm{X})\,, \qquad (\mathscr{K}f)(x,x') = \int_{\mathrm{Y}} f(x-x'+y)\chi(\frac{1}{2}\langle y, x+x'\rangle)\, d\mu_{\mathrm{Y}}(y)\,.$$

This map is an isomorphism and extends to an isomorphism between the associated spaces of tempered distributions

$$\mathscr{K} : \mathrm{S}^*(\mathrm{W}) \to \mathrm{S}^*(\mathrm{X}\times\mathrm{X})\,,$$

where $\mathrm{S}^*(\mathrm{W})$ (resp. $\mathrm{S}^*(\mathrm{X}\times\mathrm{X})$) is the space of tempered distributions on W (resp. $\mathrm{X}\times\mathrm{X}$). Moreover, we consider the map Op defined by

$$\mathrm{S}(\mathrm{X}\times\mathrm{X}) \ni K \to \mathrm{Op}(K) \in \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}(\mathrm{X}))\,, \qquad \mathrm{Op}(K)v(x) = \int_X K(x,x')v(x')d\mu_X(x')\,.$$

The map Op can be extended to $\mathrm{S}^*(\mathrm{X}\times\mathrm{X})$ and the corresponding map

$$\mathrm{Op} : \mathrm{S}^*(\mathrm{W}) \to \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}^*(\mathrm{X}))$$

is an isomorphism.
Let $\omega : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}^*(\mathrm{X}))$ be the map given by

$$\omega = \mathrm{Op} \circ \mathscr{K} \circ \mathrm{T}\,.$$

As proved in [1, Section 4], we get that for every $\widetilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ and $v \in \mathrm{S(X)}$, $\omega(\widetilde{g})v \in \mathrm{S(X)}$ and that $\omega(\widetilde{gh}) = \omega(\widetilde{g}) \circ \omega(\widetilde{h})$ for every $\widetilde{g}, \widetilde{h} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$. The operator $\omega(\widetilde{g}) \in \mathrm{Hom}(\mathrm{S(X)}, \mathrm{S(X)})$ can be extended to $\mathrm{L}^2(\mathrm{X})$ by

$$\omega(\widetilde{g})\phi = \lim_{\substack{\|\phi - v\|_2 \to 0 \\ v \in \mathrm{S(X)}}} \omega(\widetilde{g})v\,, \qquad \left(\phi \in \mathrm{L}^2(\mathrm{X})\right).$$

Theorem 2.1. *For every* $\widetilde{g} \in \widetilde{\mathrm{S}}(\mathrm{W})$ *and* $\phi \in \mathrm{L}^2(\mathrm{X})$*, the map*

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \widetilde{g} \to \omega(\widetilde{g})\phi \in \mathrm{L}^2(\mathrm{X})\,,$$

is well-defined and continuous. Moreover, $\omega(\widetilde{g}) \in \mathrm{U}(\mathrm{L}^2(\mathrm{X}))$*, i.e.* ω *is a faithful unitary representation of* $\widetilde{\mathrm{Sp}}(\mathrm{W})$*, and for every* $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$*, we get:*

$$\int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Theta(\widetilde{g})\Psi(\widetilde{g})d\widetilde{g} = \mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Psi(\widetilde{g})\omega(\widetilde{g})d\widetilde{g}\,,$$

where $d\widetilde{g}$ *is a Haar measure on* $\widetilde{\mathrm{Sp}}(\mathrm{W})$*.*

In the following, we write $\mathscr{H} = \mathrm{L}^2(X)$, $\mathscr{H}^\infty = \mathrm{S}(X)$ and ω^∞ for the smooth representation on $\mathrm{S}(X)$ associated with ω.

3. Howe duality theorem

In this section, W is a vector space over $\mathbb{R}$ endowed with a non-degenerate, skew-symmetric, bilinear form $\langle \cdot, \cdot \rangle$ and $(\mathrm{G}, \mathrm{G}')$ is an irreducible reductive dual pair in $\mathrm{Sp}(\mathrm{W})$.

Lemma 3.1. *Let* $\widetilde{\mathrm{G}} = \mathrm{pr}^{-1}(\mathrm{G})$ *and* $\widetilde{\mathrm{G}'} = \mathrm{pr}^{-1}(\mathrm{G}')$ *be respectively the preimages of* G *and* G′ *in the metaplectic group* $\widetilde{\mathrm{Sp}}(\mathrm{W})$*. Then* $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}'})$ *is a dual pair in* $\widetilde{\mathrm{Sp}}(\mathrm{W})$*.*

Let $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ be the set of equivalence classes of irreducible admissible Hilbert representations of $\widetilde{\mathrm{G}}$ which are infinitesimally equivalent to a quotient of $\mathscr{H}^\infty$ by a closed $\omega^\infty(\widetilde{\mathrm{G}})$-invariant subspace.
The identity maps of $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}'}$ induce a homomorphism $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}'} \to \widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}'} \subset \widetilde{\mathrm{Sp}}(\mathrm{W})$ that allows us to consider $\omega^\infty_{|\widetilde{\mathrm{G}}\cdot\widetilde{\mathrm{G}'}}$ as a representation of $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}'}$.

Theorem 3.2 (Howe's duality theorem, [7])**.** $\mathscr{R}(\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}'})$ *is the graph of a bijection between* $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ *and* $\mathscr{R}(\widetilde{\mathrm{G}'}, \omega)$*.*

More precisely, if $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$, we denote by $\mathrm{N}(\Pi)$ the intersection of all the closed $\widetilde{\mathrm{G}}$-invariant subspaces $\mathscr{N}$ such that $\Pi \approx \mathscr{H}^\infty/\mathscr{N}$. Then, the space $\mathscr{H}(\Pi) = \mathscr{H}^\infty/\mathrm{N}(\Pi)$ is a $\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}'}$-module; more precisely, $\mathscr{H}(\Pi) = \Pi \otimes \Pi'_1$, where Π'_1 is a $\widetilde{\mathrm{G}'}$-module, not irreducible in general, but Howe's duality theorem says that there exists a unique irreducible quotient Π' of Π'_1 with $\Pi' \in \mathscr{R}(\widetilde{\mathrm{G}'}, \omega)$ and $\Pi \otimes \Pi' \in \mathscr{R}(\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}'}, \omega)$. We will denote by $\theta : \mathscr{R}(\widetilde{\mathrm{G}}, \omega) \to \mathscr{R}(\widetilde{\mathrm{G}'}, \omega)$ the corresponding bijection.

4. Dual pairs with G compact

Let $(\mathrm{W}, \langle \cdot, \cdot \rangle)$ be a real symplectic space, $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ be a complete polarization of W and $(\mathrm{G}, \mathrm{G}')$ be an irreducible dual pair in $\mathrm{Sp}(\mathrm{W})$ with G compact. By classification (see [13]), there are three cases for G′, namely $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{U}(p, q, \mathbb{C})$ and $\mathrm{O}^*(2n, \mathbb{H})$.

Remark 4.1. If one member is compact, the situation is a bit different. In this context, (Π, V_Π) is a subrepresentation of $(\omega^\infty, \mathscr{H}^\infty)$ and in particular, its isotypic component $\mathrm{V}(\Pi)$ is a closed subspace of $\mathrm{S}(X)$. Moreover, $\widetilde{\mathrm{G}}'$ acts on $\mathrm{V}(\Pi)$ and we have $\mathrm{V}(\Pi) = \Pi \otimes \Pi'$, where Π' is an irreducible admissible representation of $\widetilde{\mathrm{G}}'$. Finally, we have:

$$\omega^\infty = \bigoplus_{\Pi \in \hat{\mathrm{G}}_\omega} \Pi \otimes \Pi' \,, \tag{2}$$

where $\hat{\mathrm{G}}_\omega$ is the set of irreducible unitary representation $(\Pi, \mathscr{H}_\Pi)$ of $\widetilde{\mathrm{G}}$ such that $\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\Pi, \omega^\infty) \neq \{0\}$, and where the sum is not an algebraic sum but the closure of the algebraic sum with respect to the topology of $\mathrm{S}(X)$.

Let $(\Pi, \mathscr{H}_\Pi)$ be a representation of $\widetilde{\mathrm{G}}$ appearing in the correspondence and Π' be the corresponding representation of $\widetilde{\mathrm{G}'}$. Denote by $\mathscr{P}_\Pi$ the projection onto the Π-isotypic component $\mathscr{H}(\Pi)$, where

$$\mathscr{H}(\Pi) = \overline{\left\{T(\mathscr{H}_\Pi), T \in \mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathscr{H}_\Pi, \mathscr{H}^\infty)\right\}} \,.$$

Using Equation (2), we get that $\mathscr{H}(\Pi) = \mathscr{H}(\Pi') = \mathscr{H}(\Pi \otimes \Pi')$. Using the classical representation theory of compact groups, we get:

$$\mathscr{P}_\Pi = d_\Pi \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})} \omega(\widetilde{g}) d\widetilde{g} = \omega(d_\Pi \overline{\Theta_\Pi}) \,,$$

and with the previous remark, $\mathscr{P}_\Pi = \mathscr{P}_{\Pi'} = \mathscr{P}_{\Pi \otimes \Pi'}$.
In particular, for all $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$, we have

$$\Theta_{\Pi'}(\Psi) = \frac{1}{d_\Pi} \mathrm{tr}\,(\mathscr{P}_{\Pi \otimes \Pi'} \omega(\Psi)) = \mathrm{tr} \int_{\widetilde{\mathrm{G}}} \int_{\widetilde{\mathrm{G}'}} \overline{\Theta_\Pi(\widetilde{g})} \Psi(\widetilde{g'}) \omega(\widetilde{g}\widetilde{g'}) d\widetilde{g'} d\widetilde{g} \,.$$

<u>Problem:</u> How can we compute this trace? The function Θ introduced in the definition of ω turns out to be the character of the metaplectic representation, but this function is not continuous. So, the integral

$$\int_{\widetilde{\mathrm{G}}} \int_{\widetilde{\mathrm{G}'}} \overline{\Theta_\Pi(\widetilde{g})} \Psi(\widetilde{g'}) \Theta(\widetilde{g}\widetilde{g'}) d\widetilde{g'} d\widetilde{g}$$

(obtained by formally taking traces under the integral sign) is not necessary well defined.

<u>Solution :</u> This problem can be avoided using the Howe oscillator semigroup $\widetilde{\mathrm{Sp}}(\mathrm{W}_\mathbb{C})^{++}$ introduced by Howe in [7]. This semigroup contains the metaplectic group in its closure and the function Θ has a holomorphic extension to $\widetilde{\mathrm{Sp}}(\mathrm{W}_\mathbb{C})^{++}$. Moreover,

$$\widetilde{\mathrm{Sp}}(\mathrm{W}_\mathbb{C})^{++} \widetilde{\mathrm{Sp}(\mathrm{W})} \subseteq \widetilde{\mathrm{Sp}}(\mathrm{W}_\mathbb{C})^{++} \,,$$

and for all $\widetilde{h} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, we have

$$\Theta(\widetilde{h}) = \lim_{\substack{\widetilde{p} \in \widetilde{\mathrm{Sp}}(\mathrm{W}_\mathbb{C})^{++} \\ \widetilde{g} \to 1}} \Theta(\widetilde{g}\widetilde{h}) \,.$$

Proposition 4.2.

$$\Theta_{\Pi'}(\Psi) = \lim_{\substack{\widetilde{p} \to 1 \\ \widetilde{p} \in \widetilde{\mathrm{Sp}}(\mathrm{W}_\mathbb{C})^{++}}} \int_{\widetilde{\mathrm{G}}} \int_{\widetilde{\mathrm{G}'}} \overline{\Theta_\Pi(\widetilde{g})} \Psi(\widetilde{g'}) \Theta(\widetilde{p}\widetilde{g}\widetilde{g'}) d\widetilde{g'} d\widetilde{g} \,, \tag{3}$$

i.e. for every $\widetilde{g}' \in \widetilde{\mathrm{G}}'^{\mathrm{reg}}$ *we have:*

(4)
$$\Theta_{\Pi'}(\widetilde{g}') = \lim_{\substack{\widetilde{p}\to 1 \\ \widetilde{p}\in\widetilde{\mathrm{Sp}}(\mathrm{W}_{\mathbb{C}})^{++}}} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_{\Pi}(\widetilde{g})}\Theta(\widetilde{p}\widetilde{g}\widetilde{g}')d\widetilde{g}\,.$$

Proof. See [11, Theorem 4.3]. □

Let's now make some explicit computations for the dual pair $(\mathrm{G} = \mathrm{U}(1,\mathbb{C}), \mathrm{G}' = \mathrm{U}(p,q,\mathbb{C}))$. In this case, G is abelian. Then, every irreducible representation of G is a character. More precisely, they are parametrized by an element $k \in \mathbb{Z}$

(5)
$$\Pi_k(x = e^{it}) = (e^{it})^k = e^{ikt}\,.$$

How can we compute this integral?

(1) Denote by $\mathrm{F}_{\widetilde{p},\widetilde{g}'} : \widetilde{\mathrm{G}} \to \mathbb{C}$ given in the equation (4):

(6)
$$\mathrm{F}_{\widetilde{p},\widetilde{g}'}(\widetilde{g}) = \overline{\Theta_{\Pi}(\widetilde{g})}\Theta(\widetilde{p}\widetilde{g}\widetilde{g}')\,.$$

Using classical results of differential geometry, we obtain:

$$\int_{\widetilde{\mathrm{G}}} \mathrm{F}_{\widetilde{p},\widetilde{g}'}(\widetilde{g})\,d\widetilde{g} = 2\int_{\mathrm{G}} \mathrm{H}_{\widetilde{p},\widetilde{g}'}(g)de\,.$$

where $\mathrm{H}_{\widetilde{p},\widetilde{g}'}$ is defined as $\mathrm{H}_{\widetilde{p},\widetilde{g}'}(\mathrm{pr}(\widetilde{g})) = \mathrm{F}_{\widetilde{p},\widetilde{g}'}(\widetilde{g})$ (we show that this function is well defined)

(2) Using the Weyl integration formula, we obtain an integral over T (because the function $\mathrm{H}_{\widetilde{p},\widetilde{g}'}$ is central),

(3) Let T' the compact Cartan subgroup of G' and let $(\mathrm{T}'_{\mathbb{C}})^{++}$ defined by

(7)
$$(\mathrm{T}'_{\mathbb{C}})^{++} = \{\mathrm{diag}(t_1,\ldots,t_{p+q})\,;\ |t_i| < 1 \text{ pour } 1 \leq i \leq p, |t_i| > 1 \text{ pour } p < i \leq p+q\}$$

Lemma 4.3. *For all* $\widetilde{t} \in \widetilde{\mathrm{U}}(1,\mathbb{C})$ *and* $\widetilde{t}' \in (\widetilde{\mathrm{T}}'_{\mathbb{C}})^{++}$*, we get:*

(8)
$$\Theta(\widetilde{tt'}) = \frac{(-1)^q t_1^{\frac{p+q}{2}} \left(\prod_{a=1}^{p+q} t'_a\right)^{\frac{1}{2}}}{\prod_{a=1}^{p}(t_1 - t'_a) \prod_{a=p+1}^{p+q}\left(t_1 - \frac{1}{t'_a}\right)}$$

Proof. See [11, Proposition 6.1]. □

Remark 4.4 (Where does this formula come from?). As explained in [1], if $g \in \mathrm{Sp}(\mathrm{W})$ is regular, the point $(g,\xi) \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ satisfy $\xi^2 = \det(i(g-1))^{-1}$. Let $\mathrm{U}(1,\mathbb{C})$ a maximal compact subgroup of $\mathrm{Sp}(2,\mathbb{R})$. Then, for every $t' = e^{i\theta} \in \mathrm{U}(1,\mathbb{C}), \theta \in 2\pi\mathbb{Z}$, we get

$$\begin{aligned}
\det(i(t-1)) &= \det\left(\begin{pmatrix} i\cos(\theta) - i & -i\sin(\theta) \\ i\sin(\theta) & i\cos(\theta) - i\end{pmatrix}\right) = (i\cos(\theta) - i)^2 - \sin(\theta)^2 \\
&= 2\cos(\theta) - 2 = -4\sin^2(\frac{\theta}{2}) = (e^{\frac{i\theta}{2}} - e^{-\frac{i\theta}{2}})^2 \\
&= \frac{(1 - e^{i\theta})^2}{(e^{\frac{i\theta}{2}})^2}
\end{aligned}$$

avec finally, by taking the inverse, we get $\xi = \pm\frac{\sqrt{(t')}}{1-t'}$. It's what we got, in terms of distributions, in Equation (8), with $p = 0, q = 1$, and $t = 1$.

Finally, we get:

Theorem 4.5. *For all $\widetilde{t'} \in \widetilde{\mathrm{T}'}^{\mathrm{reg}}$, we get:*

$$\Theta_{\Pi'_k}(\widetilde{t'}) = \pm \begin{cases} \prod_{i=1}^{p+q} t_i^{\frac{1}{2}} \sum_{h=1}^{p} \frac{t_h^{p-(k+1)}}{\prod_{h\neq j}(t_h - t_j)} & \text{if } p \geqslant k+1 \\ -\prod_{i=1}^{p+q} t_i^{\frac{1}{2}} \sum_{h=p+1}^{p+q} \frac{t_h^{p-(k+1)}}{\prod_{h\neq j}(t_h - t_j)} & \text{if } p < k+1 \end{cases} \tag{9}$$

References

[1] Anne-Marie Aubert and Tomasz Przebinda. A reverse engineering approach to the Weil representation. *Cent. Eur. J. Math.*, 12(10):1500–1585, 2014.

[2] Thomas J. Enright. Analogues of Kostant's u-cohomology formulas for unitary highest weight modules. *J. Reine Angew. Math.*, 392:27–36, 1988.

[3] Harish-Chandra. Representations of a semisimple Lie group on a Banach space. I. *Trans. Amer. Math. Soc.*, 75:185–243, 1953.

[4] Harish-Chandra. Representations of semisimple Lie groups. III. *Trans. Amer. Math. Soc.*, 76:234–253, 1954.

[5] Harish-Chandra. Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119:457–508, 1965.

[6] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. *Acta Math.*, 116:1–111, 1966.

[7] Roger Howe. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535–552, 1989.

[8] A. A. Kirillov. The method of orbits in the theory of unitary representations of Lie groups. *Funkcional. Anal. i Priložen.*, 2(1):96–98, 1968.

[9] Anthony W. Knapp. *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.

[10] Jian-Shu Li. Minimal representations & reductive dual pairs. In *Representation theory of Lie groups (Park City, UT, 1998)*, volume 8 of *IAS/Park City Math. Ser.*, pages 293–340. Amer. Math. Soc., Providence, RI, 2000.

[11] Allan Merino. Transfer of characters in the theta correspondence with one compact member. *J. Lie Theory*, 30(4):997–1026, 2020.

[12] Wulf Rossmann. Kirillov's character formula for reductive Lie groups. *Invent. Math.*, 48(3):207–220, 1978.

[13] Matthias Schmidt. Classification and partial ordering of reductive Howe dual pairs of classical Lie groups. *J. Geom. Phys.*, 29(4):283–318, 1999.