TRANSFER OF CHARACTERS FOR DISCRETE SERIES REPRESENTATIONS OF THE UNITARY GROUPS IN THE EQUAL RANK CASE VIA THE CAUCHY-HARISH-CHANDRA INTEGRAL

ALLAN MERINO

Abstract. Here are the notes of a talk I gave at NUS on February 2021. The goal was to recall the construction of the Cauchy–Harish-Chandra integral, a conjecture concerning the transfer of characters in the theta correspondence and the mains ideas of my paper [13] on the proof of this conjecture for a dual pair of unitary groups of same ranks starting from a discrete series representation.

Contents

1. Harish-Chandra's Character Theory

We start this section by fixing few notations.

Notation 1.1.
- G real reductive Lie group (connected),
- $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ its complexification,
- $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ enveloping algebra, $Z(\mathscr{U}(\mathfrak{g}_{\mathbb{C}}))$ center of $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$,
- D(G): set of differential operators on G,
- $\tau : G \curvearrowright D(G)$ the action of G on D(G) given by

$$(\tau(g)D) f = L_g \circ D\left(f \circ L_{g^{-1}}\right), \qquad (g \in G, f \in \mathscr{C}_c^{\infty}(G), D \in D(G)) \ ,$$

- $D_G(G) = \{D \in D(G), \tau(g) D = D, g \in G\}$ the set of left invariant differential operators (it is well-known that $D_G(G) \approx \mathscr{U}(\mathfrak{g}_{\mathbb{C}})$),
- $D^G_G(G)$: bi-invariant differential operators ($D^G_G(G) \approx Z(\mathscr{U}(\mathfrak{g}_{\mathbb{C}}))$)
- $\mathscr{D}'(G)$ set of distributions on G, i.e. the set of continuous linear forms on $\mathscr{C}_c^{\infty}(G)$,

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- $\delta : \mathrm{G} \curvearrowright \mathscr{C}_c^\infty(\mathrm{G})$ the action of G on $\mathscr{C}_c^\infty(\mathrm{G})$ given by
$$(\delta(g)f)\, x = f^g(x), \qquad (g, x \in \mathrm{G}, f \in \mathscr{C}_c^\infty(\mathrm{G}))\ ,$$
where $f^g(x) = f(gxg^{-1})$. The action δ can be extended can be extended to $\mathscr{D}'(\mathrm{G})$ by
$$(\delta(g)T)\, f = T(f^g), \qquad (g \in \mathrm{G}, T \in \mathscr{D}'(\mathrm{G}), f \in \mathscr{C}_c^\infty(\mathrm{G}))\,,$$
- $\mathscr{D}'(\mathrm{G})^{\mathrm{G}} = \{\mathrm{D} \in \mathscr{D}'(\mathrm{G}), \delta(g)\mathrm{D} = \mathrm{D}\}$ the set of G-invariant distributions.

Definition 1.2. We say that $T \in \mathscr{D}'(\mathrm{G})$ is an eigendistribution if there exists $\chi_T : \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G}) \to \mathbb{C}$ an homomorphism of algebras such that $\mathrm{D}(T) = \chi_T(\mathrm{D})T$ for every $\mathrm{D} \in \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G})$.

Notation 1.3. Eigen(G) = {eigendistributions on G}.

Theorem 1.4. *For every* G*-invariant eigendistribution T on* G*, there exists a locally integrable function f_T on* G*, analytic on* $\mathrm{G}^{\mathrm{reg}}$*, such that $T = T_{f_T}$, i.e. for every function $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$,*
$$T(\Psi) = \int_{\mathrm{G}} f_T(g)\Psi(g)dg\,,$$
where dg is a Haar measure on G*.*

We apply the previous theorem to some particular representations of G. Let $(\Pi, \mathscr{H})$ be an irreducible quasi-simple representation and let $\mathscr{H}^\infty$ be the space of smooth vectors. By quasi-simple we mean that both Z(G) and $\mathrm{Z}(\mathscr{U}(\mathfrak{g}_{\mathbb{C}}))$ act by a character, where
$$\mathrm{Z}(\mathrm{G}) \curvearrowright \mathscr{H} \supseteq \mathscr{H}^\infty \curvearrowleft \mathrm{Z}(\mathscr{U}(\mathfrak{g}_{\mathbb{C}}))\,.$$
The character corresponding to the action $\mathrm{Z}(\mathscr{U}(\mathfrak{g}_{\mathbb{C}})) \curvearrowright \mathscr{H}^\infty$ will be denoted by χ_Π. For every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, we define
$$\Pi(\Psi) = \int_{\mathrm{G}} \Psi(g)\Pi(g)dg\,.$$
The operator $\Pi(\Psi)$ is well-defined and bounded.

Theorem 1.5. [5, Section 5]
For every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, $\Pi(\Psi)$ is a trace class operator. Moreover, the map:
$$\Theta_\Pi : \mathscr{C}_c^\infty(\mathrm{G}) \ni \Psi \to \mathrm{tr}\,(\Pi(\Psi)) \in \mathbb{C}$$
is a distribution (in the sense of Laurent Schwartz).

Because $(\Pi, \mathscr{H})$ is quasi-simple, there exists $\chi_\Pi : \mathrm{Z}(\mathscr{U}(\mathfrak{g}_{\mathbb{C}})) \to \mathbb{C}$ such that $z\Theta_\Pi = \chi_\Pi(z)\Theta_\Pi$ for every $z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}_{\mathbb{C}}))$. In particular, there exists $\Theta_\Pi \in \mathscr{L}^1_{\mathrm{loc}}(\mathrm{G})$ such that
$$\Theta_\Pi(\Psi) = \int_{\mathrm{G}} \Theta_\Pi(g)\Psi(g)dg$$
for every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$.

Remark 1.6. In some cases, the function Θ_Π is well-known:
- G compact (Hermann Weyl, ~ 1925),
- Π is a discrete series representation:
  1. Harish-Chandra: value of Θ_Π on the compact Cartan subgroup (see [6] and [7]),
  2. Schmid-Hecht: value of Θ_Π on the other Cartan subgroups for holomorphic discrete series representations (see [8])
- $(\Pi, \mathscr{H})$ is an irreducible unitary highest weight module (Enright, see [4]).

2. Howe's duality theorem

Let χ be the character of $\mathbb{R}$ given by $\chi(r) = e^{2i\pi r}$ and $(\mathrm{W}, \langle\cdot,\cdot\rangle)$ be a symplectic vector space over $\mathbb{R}$. We denote by $\mathrm{Sp}(\mathrm{W})$ the group of isometries corresponding to $(\mathrm{W}, \langle\cdot,\cdot\rangle)$, i.e.

$$\mathrm{Sp}(\mathrm{W}) = \{g \in \mathrm{GL}(\mathrm{W}), \langle g(u), g(v)\rangle = \langle u, v\rangle, u, v \in \mathrm{W}\},$$

and by $\mathfrak{sp}(\mathrm{W})$ the Lie algebra of $\mathrm{Sp}(\mathrm{W})$.
We recall quickly a construction of a (connected) double cover $\widetilde{\mathrm{Sp}}(\mathrm{W})$ of $\mathrm{Sp}(\mathrm{W})$: the metaplectic group. Let J be an element of $\mathfrak{sp}(\mathrm{W})$ satisfying $\mathrm{J}^2 = -\mathrm{Id}$ and such that the symmetric, bilinear form $(\cdot,\cdot)$ on W given by $(u, v) = \langle \mathrm{J}(u), v\rangle$ is positive definite.
For every $g \in \mathrm{Sp}(\mathrm{W})$, we denote by J_g the endomorphism of W given by $\mathrm{J}_g = \mathrm{J}^{-1}(g-1)$. One can easily check that the restriction of J_g to $\mathrm{J}_g(\mathrm{W})$ defines an invertible element. In particular, the following determinant $\det(\mathrm{J}_g)^{-1}_{\mathrm{J}_g(\mathrm{W})}$ makes sense.
We denote by $\widetilde{\mathrm{Sp}}(\mathrm{W})$ the following subset of $\mathrm{Sp}(\mathrm{W}) \times \mathbb{C}^*$ given by

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) = \left\{(g, \xi) \in \mathrm{Sp}(\mathrm{W}) \times \mathbb{C}^*, \xi^2 = \det(\mathrm{J}_g)^{-1}_{\mathrm{J}_g(\mathrm{W})}\right\}.$$

We define a multiplication on $\widetilde{\mathrm{Sp}}(\mathrm{W})$ by

$$(g_1, \xi_1) \cdot (g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 \mathrm{C}(g_1, g_2)), \qquad (g_1, g_2 \in \mathrm{Sp}(\mathrm{W}), \xi_1, \xi_2 \in \mathbb{C}^*),$$

where $\mathrm{C} : \mathrm{Sp}(\mathrm{W}) \times \mathrm{Sp}(\mathrm{W}) \to \mathbb{C}^*$ is a cocycle of $\mathrm{Sp}(\mathrm{W})$ defined in [1, Section 4] whose absolute value satisfies

$$|\mathrm{C}(g_1, g_2)| = \sqrt{\left|\frac{\det(\mathrm{J}_{g_1})_{\mathrm{J}_{g_1}(\mathrm{W})} \det(\mathrm{J}_{g_2})_{\mathrm{J}_{g_2}(\mathrm{W})}}{\det(\mathrm{J}_{g_1 g_2})_{\mathrm{J}_{g_1 g_2}(\mathrm{W})}}\right|}.$$

The group $\widetilde{\mathrm{Sp}}(\mathrm{W})$ is a double cover or $\mathrm{Sp}(\mathrm{W})$ whose covering map π is given by $\pi(g, \xi) = g$.
In the next sections, we will be interested in a special representation of $\widetilde{\mathrm{Sp}}(\mathrm{W})$: the Weil representation. We quickly recall a construction of this representation (for more details, see [1]). We denote by $\mathrm{T} : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W})$ the embedding of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ onto the space of tempered distributions on W given by

$$\mathrm{T}(\tilde{g}) = \Theta(\tilde{g}) \chi_{c(g)} \mu_{(g-1)\mathrm{W}}, \qquad (\tilde{g} = (g, \xi)), \tag{1}$$

where $\Theta((g, \xi)) = \xi$, $\mu_{(g-1)\mathrm{W}}$ is the Lebesgue measure on $(g-1)\mathrm{W}$ normalized so that the volume of the unit cube with respect to the form $(\cdot,\cdot)$ is 1, and $\chi_{c(g)}$ is the function on $(g-1)\mathrm{W}$ such that

$$\chi_{c(g)}((g-1)w) = \chi\left(\frac{1}{4}\langle (g+1)w, (g-1)w\rangle\right), \qquad (w \in \mathrm{W}).$$

Let $\mathrm{W} = \mathrm{X} \oplus \mathrm{Y}$ be a complete polarization of W. We denote by $\mathscr{K} : \mathrm{S}(\mathrm{W}) \to \mathrm{S}(\mathrm{X} \times \mathrm{X})$ the map given by

$$(\mathscr{K} f)(x, x') = \int_{\mathrm{Y}} f(x - x' + y) \chi\left(\frac{1}{2}\langle y, x + x'\rangle\right) dy, \qquad (f \in \mathrm{S}(\mathrm{W}), x, x' \in \mathrm{X}).$$

This map is an isomorphism and can be extended to an isomorphism

$$\mathscr{K} : \mathrm{S}^*(\mathrm{W}) \to \mathrm{S}^*(\mathrm{X} \times \mathrm{X}).$$

Similarly, let $\mathrm{Op} : \mathrm{S}(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}(\mathrm{X}))$ the map defined by

$$\mathrm{Op}(\mathrm{K})v(x) = \int_{\mathrm{X}} \mathrm{K}(x, x') v(x') dx', \qquad (\mathrm{K} \in \mathrm{S}(\mathrm{X} \times \mathrm{X}), x \in \mathrm{X}).$$

This map is not bijective (indeed, the identity operator of Hom(S(X), S(X)) is not a Kernel operator), but the corresponding extension

$$\mathrm{Op} : \mathrm{S}^*(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}^*(\mathrm{X}))$$

is an isomorphism of linear topological vector spaces. This result is known as the Schwartz's Kernel theorem.

Lemma 2.1. *Let* $\omega = \mathrm{Op} \circ \mathscr{K} \circ \mathrm{T} : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W}) \to \mathrm{S}^*(\mathrm{X} \times \mathrm{X}) \to \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}^*(\mathrm{X}))$. *Then, for every* $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$ *and* $v \in \mathrm{S}(\mathrm{X})$, $\omega(\tilde{g})v \in \mathrm{S}(\mathrm{X})$.

In particular, for every $\tilde{g} \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, $\omega(\tilde{g}) \in \mathrm{Hom}(\mathrm{S}(\mathrm{X}), \mathrm{S}(\mathrm{X}))$. Moreover, it can be extended to L^2 (X) as

$$\omega(\tilde{g})\phi = \lim_{\substack{\|v-\phi\|_2 \to 0 \\ v \in \mathrm{S}(\mathrm{X})}} \omega(\tilde{g})v, \qquad \left(\phi \in \mathrm{L}^2(\mathrm{X})\right).$$

One can easy check that $\omega(\tilde{g})\phi \in \mathrm{L}^2(\mathrm{X})$ and the corresponding operator $\omega(\tilde{g})$ is unitary. Moreover, for every $\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}(\mathrm{W})$, $\omega(\tilde{g}_1\tilde{g}_2) = \omega(\tilde{g}_1) \circ \omega(\tilde{g}_2)$ and for every $\phi \in \mathrm{L}^2(\mathrm{X})$, the map

$$\widetilde{\mathrm{Sp}}(\mathrm{W}) \ni \tilde{g} \to \omega(\tilde{g})\phi \in \mathrm{L}^2(\mathrm{X})$$

is continuous. In particular,

$$\omega : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{U}\left(\mathrm{L}^2(\mathrm{X})\right)$$

is a faithful unitary representation of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ (see [1, Theorem 4.27]). Moreover, for every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$,

$$\mathrm{tr} \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Psi(\tilde{g})\omega(\tilde{g})d\tilde{g} = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Psi(\tilde{g})\Theta(\tilde{g})d\tilde{g},$$

i.e. Θ is the character of ω.

Definition 2.2. A dual pair in Sp(W) is a pair of subgroups $(\mathrm{G}, \mathrm{G}')$ of Sp(W) such that $\mathrm{G} = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}\mathrm{G}'$ and $\mathrm{G}' = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}\mathrm{G}'$.
The pair is said to be:

- reductive if $\mathrm{G}, \mathrm{G}' \curvearrowright \mathrm{W}$ reductively,
- irreducible if we cannot find an orthogonal decomposition $\mathrm{W} = \mathrm{W}_1 \oplus \mathrm{W}_2$ where both W_1 and W_2 are $\mathrm{G} \cdot \mathrm{G}'$-invariant.

The irreducible reductive dual pairs in Sp(W) had been classified by Howe in [9].

Example 2.3. Let $(\mathrm{V}, (\cdot, \cdot))$ be a symplectic space over $\mathbb{R}$ and $(\mathrm{V}', (\cdot, \cdot)')$ be an orthogonal space over $\mathbb{R}$. Then, $(\mathrm{V} \otimes_{\mathbb{R}} \mathrm{V}', (\cdot, \cdot) \otimes (\cdot, \cdot)')$ is a symplectic space and

$$(\mathrm{Sp}(\mathrm{V}, (\cdot, \cdot)), \mathrm{O}(\mathrm{V}', (\cdot, \cdot)')) \subseteq \mathrm{Sp}\left(\mathrm{V} \otimes_{\mathbb{R}} \mathrm{V}', (\cdot, \cdot) \otimes (\cdot, \cdot)'\right)$$

is an irreducible reductive dual pair.
Similarly, let $(\mathrm{V}, (\cdot, \cdot))$ be an hermitian space over $\mathbb{C}$ and $(\mathrm{V}', (\cdot, \cdot)')$ be a skew-hermitian space over $\mathbb{C}$. Then, $(\mathrm{V} \otimes_{\mathbb{C}} \mathrm{V}', (\cdot, \cdot) \otimes (\cdot, \cdot)')$ is a skew-hermitian space and then, $((\mathrm{V} \otimes_{\mathbb{C}} \mathrm{V}')_{\mathbb{R}}, \mathrm{Im}((\cdot, \cdot) \otimes (\cdot, \cdot)'))$ is a real symplectic space. In particular,

$$(\mathrm{U}(\mathrm{V}, (\cdot, \cdot)), \mathrm{U}(\mathrm{V}', (\cdot, \cdot)')) \subseteq \mathrm{Sp}\left((\mathrm{V} \otimes_{\mathbb{C}} \mathrm{V}')_{\mathbb{R}}, \mathrm{Im}((\cdot, \cdot) \otimes (\cdot, \cdot)')\right)$$

is a dual pair, i.e. $(\mathrm{U}(p, q), \mathrm{U}(r, s)) \subseteq \mathrm{Sp}(2(p+q)(r+s), \mathbb{R})$ is a dual pair.

Lemma 2.4. *Let* $(\mathrm{G}, \mathrm{G}')$ *be an irreducible reductive dual pair in* Sp(W). *Then,* $(\widetilde{\mathrm{G}} = \pi^{-1}(\mathrm{G}), \widetilde{\mathrm{G}'} = \pi^{-1}(\mathrm{G}'))$ *is a dual pair in* $\widetilde{\mathrm{Sp}}(\mathrm{W})$.

Notation 2.5. We denote by $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ the set of infinitesimal equivalence classes of irreducible admissible representations $(\Pi, \mathscr{H})$ of $\widetilde{\mathrm{G}}$ which can be realized as a quotient of $\mathscr{H}^{\infty}$ by a closed $\omega^{\infty}(\widetilde{\mathrm{G}})$-invariant subspace.

Theorem 2.6. [10, Theorem 1]
$\mathscr{R}(\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}'}, \omega)$ is the graph of a bijection between $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ and $\mathscr{R}(\widetilde{\mathrm{G}'}, \omega)$.

More precisely, for every $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$, there exists a $\omega^{\infty}(\widetilde{\mathrm{G}})$-invariant subspace $\mathscr{N}_{\Pi}$ such that $\Pi \sim \mathscr{H}^{\infty} / \mathscr{N}_{\Pi}$. The space $\mathscr{N}_{\Pi}$ is not unique. Let $\mathscr{H}(\Pi)$ be the maximal quotient of Π, i.e.

$$\mathscr{H}(\Pi) = \mathscr{H}^{\infty} / \left(\bigcap_{\substack{\mathscr{N} \text{ is } \omega^{\infty}(\widetilde{\mathrm{G}})-invariant \\ \Pi \sim \mathscr{H}^{\infty}/\mathscr{N}}} \mathscr{N} \right).$$

Obviously, $\widetilde{\mathrm{G}}$ acts on $\mathscr{H}(\Pi)$ and one can easily check that $\widetilde{\mathrm{G}'}$ acts on $\mathscr{H}(\Pi)$. In particular, there exists a $\widetilde{\mathrm{G}'}$-module Π'_1 (not irreducible in general) such that $\mathscr{H}(\Pi) = \Pi \otimes \Pi'_1$. Howe's duality theorem says that the representation Π'_1 has a unique irreducible quotient Π' so that $\Pi \otimes \Pi' \in \mathscr{R}(\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}'}, \omega)$.
We denote by $\theta : \mathscr{R}(\widetilde{\mathrm{G}}, \omega) \ni \Pi \to \Pi' \in \mathscr{R}(\widetilde{\mathrm{G}'}, \omega)$ the corresponding bijection.

3. Cauchy–Harish-Chandra integral

Let $\mathrm{T} : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W})$ the map defined in Equation (1). The map T can be extended to $\mathscr{C}_c^{\infty}(\widetilde{\mathrm{Sp}}(\mathrm{W}))$ as

$$\mathrm{T}(\Psi) = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Psi(\tilde{g}) \mathrm{T}(\tilde{g}) d\tilde{g}, \qquad \left(\Psi \in \mathscr{C}_c^{\infty}(\widetilde{\mathrm{Sp}}(\mathrm{W})) \right).$$

Lemma 3.1. *For every $\Psi \in \mathscr{C}_c^{\infty}(\widetilde{\mathrm{Sp}}(\mathrm{W}))$, $\mathrm{T}(\Psi) \in \mathrm{S}(\mathrm{W})$.*

Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair in $\mathrm{Sp}(\mathrm{W})$. Let $\mathrm{H}_1, \ldots, \mathrm{H}_n$ be a maximal set of Cartan subgroups of G. For every $i \in [|1, n|]$, we consider the decomposition of H_i of the form $\mathrm{H}_i = \mathrm{T}_i \mathrm{A}_i$, where T_i is maximal compact in H_i (A_i is called the split part of H_i).
Let $\mathrm{A}'_i = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{A}_i)$ and $\mathrm{A}''_i = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{A}'_i)$. One can easily prove that $(\mathrm{A}'_i, \mathrm{A}''_i)$ is a dual pair in $\mathrm{Sp}(\mathrm{W})$ (not irreducible in general).

Proposition 3.2. *There exists an open and dense subspace $\mathrm{W}_{\mathrm{A}''_i} \subseteq \mathrm{W}$, A'_i-invariant, such that $\mathrm{A}''_i \setminus \mathrm{W}_{\mathrm{A}''_i}$, endowed with an A'_i-invariant measure $\overline{dw}$ such that*

$$\int_{\mathrm{W}} \phi(w) dw = \int_{\mathrm{A}''_i \setminus \mathrm{W}_{\mathrm{A}''_i}} \int_{\mathrm{W}''_i} \phi(aw) da \overline{dw},$$

for every $\phi \in \mathscr{C}_c^{\infty}(\mathrm{W})$ such that $\mathrm{supp}(\phi) \subseteq \mathrm{W}_{\mathrm{A}''_i}$.

Remark 3.3. We say few worlds about the dual pair $(\mathrm{A}'_i, \mathrm{A}''_i)$ and the space $\mathrm{W}_{\mathrm{A}''_i}$. Let $\mathrm{V}_{0,i}$ be the subspace of V on which A_i acts trivially and $\mathrm{V}_{1,i} = \mathrm{V}_{0,i}^{\perp}$. The restriction of $(\cdot, \cdot)$ to $\mathrm{V}_{1,i}$ is non-degenerate and even dimensional. In particular, there exists a complete polarization of $\mathrm{V}_{1,i}$ for the form $\mathrm{V}_{1,i} = \mathrm{X}_i \oplus \mathrm{Y}_i$, where both spaces X_i and Y_i are H_i-invariant.
We look at the action of A_i on $\mathrm{V}_{1,i}$. In particular, we get:

$$\mathrm{X}_i = \mathrm{X}_i^1 \oplus \ldots \oplus \mathrm{X}_i^k, \qquad \mathrm{Y}_i = \mathrm{Y}_i^1 \oplus \ldots \oplus \mathrm{Y}_i^k,$$

where all the spaces X_i^j, $1 \leq j \leq n, 1 \leq j \leq k$, are A_i-invariant and mutually non-equivalent. In particular,

$$W = \mathrm{Hom}(V, V') = \mathrm{Hom}(V_{0,i}, V') \oplus \mathrm{Hom}(V_{1,i}, V') = \mathrm{Hom}(V_{0,i}, V') \oplus \bigoplus_{j=1}^{k} \left(\mathrm{Hom}(X_i^j, V') \oplus \mathrm{Hom}(Y_i^j, V')\right).$$

To simplify, we denote by W_j^i the subspace of W given by $\mathrm{Hom}(X_i^j, V') \oplus \mathrm{Hom}(Y_i^j, V')$ and $W_{0,i} = \mathrm{Hom}(V_{0,i}, V')$. One can easily check that:

$$A_i' = \mathrm{Sp}(W_{0,i}) \times \mathrm{GL}(\mathrm{Hom}(X_i^1, V')_{\mathbb{R}}) \times \ldots \times \mathrm{GL}(\mathrm{Hom}(X_i^k, V')_{\mathbb{R}})$$

and

$$A_i'' = O(1) \times \mathrm{GL}(1, \mathbb{R}) \times \ldots \times \mathrm{GL}(1, \mathbb{R}).$$

Moreover,

$$W_{A_i''} = (W_c \setminus \{0\}) \times \widetilde{W}_{1,i} \times \ldots \times \widetilde{W}_{n,i},$$

where $\widetilde{W}_{j,i} = \left\{(x, y) \in \mathrm{Hom}(X_i^j, V') \oplus \mathrm{Hom}(Y_i^j, V'), x \neq 0, y \neq 0\right\}$, $1 \leq j \leq k$.

For every $\Psi \in \mathscr{C}_c^\infty(\widetilde{A_i'})$, we define:

$$\mathrm{Chc}(\Psi) = \int_{A_i'' \setminus W_{A_i''}} T(\Psi)(w)\overline{dw}.$$

Lemma 3.4. Chc *defines an* $\widetilde{A_i'}$*-invariant distribution on* $\widetilde{A_i'}$.

For every $\tilde{h}_i \in \widetilde{H}_i$, we define a map $\tau_{\tilde{h}_i}$ as

$$\tau_{\tilde{h}_i} : \widetilde{G'} \ni \tilde{g}' \to \tilde{h}_i \tilde{g}' \in \widetilde{A_i'}.$$

Proposition 3.5. [16, Section 2]
For every $\tilde{h}_i \in \widetilde{H}_i^{\mathrm{reg}}$*, the pull-back* $\tau^*_{\tilde{h}_i}(\mathrm{Chc})$ *of* Chc *by* $\tau_{\tilde{h}_i}$ *is a well-defined distribution on* $\widetilde{G'}$.

Notation 3.6. For every $\tilde{h}_i \in \widetilde{H}_i^{\mathrm{reg}}$, we denote by $\mathrm{Chc}_{\tilde{h}_i}$ the distribution $\tau^*_{\tilde{h}_i}(\mathrm{Chc})$ on $\widetilde{G'}$.

There are two ways to look at $\mathrm{Chc}_{\tilde{h}_i}(\Psi)$:

(1) A family of distributions of $\widetilde{G'}$ parametrized by regular elements on the different Cartan subgroups $\widetilde{H}_i$,
(2) We fix $\Psi \in \mathscr{C}_c^\infty(\widetilde{G'})$ and get a natural $\widetilde{G}$-invariant function $\mathrm{Chc}(\Psi)$ on $\widetilde{G}^{\mathrm{reg}}$ by

$$\mathrm{Chc}(\Psi)(\tilde{h}_i) := \mathrm{Chc}_{\tilde{h}_i}(\Psi), \qquad \left(\tilde{h}_i \in \widetilde{H}_i^{\mathrm{reg}}\right).$$

Using that the function is $\widetilde{G}$-invariant, it is defined by its value on the different Cartan subgroups. One can easily check that $\mathrm{Chc}(\Psi) \in \mathscr{C}^\infty(\widetilde{G}^{\mathrm{reg}})$., i.e. we get a map:

$$\widetilde{\mathrm{Chc}} : \mathscr{C}_c^\infty(\widetilde{G'}) \to \mathscr{C}^\infty(\widetilde{G}^{\mathrm{reg}}). \tag{2}$$

Theorem 3.7. *Assume that* $\mathrm{rk}(G) \leq \mathrm{rk}(G')$. *Via* $\widetilde{\mathrm{Chc}}$, *we construct a map*

$$\mathrm{Chc}^* : \mathscr{D}'(\widetilde{G})^{\widetilde{G}} \to \mathscr{D}'(\widetilde{G'})^{\widetilde{G'}}$$

such that for every $\widetilde{\mathrm{G}}$*-invariant distribution* Θ *given by a locally integrable function* Θ *on* $\widetilde{\mathrm{G}}$*, we get:*

$$\mathrm{Chc}^*(\Theta)(\Psi) = \sum_{i=1}^{n} \frac{1}{|\mathscr{W}(\mathrm{H}_i)|} \int_{\widetilde{\mathrm{H}}_i} \Theta(\tilde{h}_i)|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}_i)^{-1})_{\mathfrak{g}/\mathfrak{h}_i}|\mathrm{Chc}_{\tilde{h}_i}(\Psi)d\tilde{h}_i\,,$$

for every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$*. Moreover,* $\mathrm{Chc}^*(\mathrm{Eigen}(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}}) \subseteq \mathrm{Eigen}(\widetilde{\mathrm{G}'})^{\widetilde{\mathrm{G}'}}$.

For the construction of Chc^*, see Appendix A (or [2]).

Conjecture 3.8. Let G_1 and G'_1 be the Zariski identity components of G and G′ respectively. Let $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ satisfying $\Theta_{\Pi|_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{G}_1}}} = 0$ if $\mathrm{G} = \mathrm{O}(\mathrm{V})$, where V is an even dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$. Then, up to a constant, $\mathrm{Chc}^*(\Theta_\Pi) = \Theta_{\Pi'_1}$ on $\widetilde{\mathrm{G}'_1}$.

Remark 3.9. The conjecture is known if:

- G compact,
- $(\mathrm{G}, \mathrm{G}')$ in the stable range starting with a unitary representation Π ([17])
- $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p, q), \mathrm{U}(r, s))$, with $p + q = r + s$ and Π a discrete series representation of $\widetilde{\mathrm{U}}(p, q)$ ([13]). In Section 7, we will quickly explain how to prove the conjecture 3.8 in this particular case.

4. How to compute the Cauchy-Harish-Chandra integral?

In this section, we will recall quickly how to compute the Cauchy-Harish-Chandra integral for unitary groups. The idea is to understand first how to compute it on the compact Cartan, and use the formula we got to compute it on the other Cartan subgroups.
Let (V, b) (resp. $(\mathrm{V}', \mathrm{b}')$) be a complex hermitian (resp. skew-hermitian) space of signature (p, q) (resp. (r, s)) and let $(\mathrm{G}, \mathrm{G}') = (\mathrm{G}(\mathrm{V}, \mathrm{b}), \mathrm{G}(\mathrm{V}', \mathrm{b}'))$ be the corresponding group of isometries of (V, b) and $(\mathrm{V}', \mathrm{b}')$. Assume that $p + q \leq r + s$ with $p \leq q$, $r \leq s$ and $p \leq r$.
Let $\mathrm{H} = \mathrm{H}(\emptyset), \mathrm{H}(\mathrm{S}_1), \ldots, \mathrm{H}(\mathrm{S}_p)$ the Cartan subgroups of G, with H compact. Up to a constant,

$$\mathrm{Chc}_{\tilde{h}}(\Psi) = \mathrm{C} \int_{\widetilde{\mathrm{G}'}} \Theta(\tilde{h}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}', \qquad \left(\tilde{h} \in \widetilde{\mathrm{H}}^{\mathrm{reg}}, \Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})\right).$$

In particular, if λ is a distribution given by a locally integrable function,

$$\int_{\widetilde{\mathrm{H}}} \lambda(\tilde{h}) \left|\Delta(\tilde{h})\right|^2 \mathrm{Chc}_{\tilde{h}}(\Psi)d\tilde{h} = \mathrm{C} \int_{\widetilde{\mathrm{H}}} \int_{\widetilde{\mathrm{G}'}} \lambda(\tilde{h}) \left|\Delta(\tilde{h})\right|^2 \Theta(\tilde{h}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}'d\tilde{h}.$$

Remark 4.1. It is well-known that every distribution can be represented as a finite sum of boundary values of analytic functions. For example, if we consider $\delta_0 \in \mathscr{D}'(\mathbb{R})$, we get:

$$\delta_0 = \frac{-1}{2i\pi} \lim_{\substack{y \to 0 \\ y>0}} \left(\frac{1}{x + iy} - \frac{1}{x - iy}\right).$$

Proposition 4.2 (Weyl Integration Formula)**.** *For every* $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$*, we get:*

$$(3) \qquad \int_{\widetilde{\mathrm{G}'}} \Psi(\tilde{g}')d\tilde{g}' = \sum_{\mathrm{S} \in \Psi_n'^{\mathrm{st}}} m_{\mathrm{S}} \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \Delta_{\Phi'}(\check{h}')\Delta_{\Psi'}(\check{h}') \int_{\mathrm{G}'/\mathrm{H}'(\mathrm{S})} \Psi(g'c(\mathrm{S})\check{p}(\check{h}')c(\mathrm{S})^{-1}g'^{-1})\overline{dg'}d\check{h}'.$$

where $\mathrm{H}' = \mathrm{H}'(\emptyset), \mathrm{H}'(\mathrm{S}_1), \ldots, \mathrm{H}'(\mathrm{S}_r)$ *is a maximal set of Cartan subgroups of* G′.

Theorem 4.3. *For every* $\check{h} \in \check{\mathrm{H}} = \check{\mathrm{H}}_{\emptyset}$ *and* $\Psi \in \mathscr{C}(\widetilde{\mathrm{G}'})$*, we get:*

$$\det^{\frac{k}{2}}(\check{h})_{\mathrm{W}^{\flat}} \Delta_{\Psi}(\check{h}) \int_{\widetilde{\mathrm{G}'}} \Theta(\check{p}(\check{h})\tilde{g}')\Psi(\tilde{g}')d\tilde{g}' =$$

$$\sum_{\sigma \in \mathscr{W}(\mathrm{H}'_{\mathbb{C}})} \sum_{\mathrm{S} \subseteq \Psi'^{\mathrm{st}}_n} \mathrm{M}_{\mathrm{S}}(\sigma) \lim_{\substack{r \in \mathrm{E}_{\sigma,\mathrm{S}} \\ r \to 1}} \int_{\check{\mathrm{H}}'_{\mathrm{S}}} \frac{\det^{-\frac{k}{2}}(\sigma^{-1}(\check{h}'))_{\mathrm{W}^{\flat}} \Delta_{\Phi'(\mathrm{Z}')}(\sigma^{-1}(\check{h}')) \Delta_{\Psi'}(\check{h}')}{\det(1 - p(h) r p(h'))_{\sigma \mathrm{W}^{\flat}}} \int_{\mathrm{G}'/\mathrm{H}'(\mathrm{S})} \Psi(g' c(\mathrm{S}) \check{p}(\check{h}') c(\mathrm{S})^{-1} g'^{-1}) \overline{dg'} d\check{h}',$$

where $\mathrm{M}_{\mathrm{S}}(\sigma) = \dfrac{(-1)^u \varepsilon(\sigma) m_{\mathrm{S}}}{|\mathscr{W}(\mathrm{Z}'_{\mathbb{C}}, \mathrm{H}'_{\mathbb{C}})|}$ *and*

$$k = \begin{cases} -1 & \text{if } r + s - p - q \in 2\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}.$$

How about Chc on the other Cartan subgroups?

Notation 4.4. For every $i \in [|1, p|]$, we denote by S_i and S'_i the subsets of Ψ^{st}_n and Ψ'^{st}_n respectively given by

$$\mathrm{S}_i = \left\{e_1 - e_{p+1}, \ldots, e_i - e_{p+i}\right\}, \qquad \mathrm{S}'_i = \left\{e'_1 - e'_{r+1}, \ldots, e'_i - e'_{r+i}\right\}.$$

- For every $i \in [|0, p|]$,we denote by $\mathrm{H}(\mathrm{S}_i)$ and $\mathrm{H}'(\mathrm{S}_i)$ the Cartan subgroups of G and G′ ,
- $\mathrm{H}(\mathrm{S}_i) = \mathrm{T}(\mathrm{S}_i)\mathrm{A}(\mathrm{S}_i)$, $\mathrm{T}(\mathrm{S}_i)$ maximal compact in $\mathrm{H}(\mathrm{S}_i)$.
- $\mathrm{V}_{0,i}$ the subspace of V on which $\mathrm{A}(\mathrm{S}_i)$ acts trivially,
- $\mathrm{V}_{1,i}$ the orthogonal complement of $\mathrm{V}_{0,i}$ in V and by $\mathrm{V}_{1,i} = \mathrm{X}_i \oplus \mathrm{Y}_i$ a complete polarization of $\mathrm{V}_{1,i}$.
- $p \leq r$, then we have a natural embedding of $\mathrm{V}_{1,i}$ into V′ such that $\mathrm{X}_i \oplus \mathrm{Y}_i$ is a complete polarization with respect to $(\cdot, \cdot)'$.
- U_i the orthogonal complement of $\mathrm{V}_{1,i}$ in V′; in particular, we get a natural embedding:

$$\mathrm{GL}(\mathrm{X}_i) \times \mathrm{G}(\mathrm{U}_i) \subseteq \mathrm{G}' = \mathrm{U}(r, s).$$

- $\mathrm{T}_1(\mathrm{S}_i)$ the maximal subgroup of $\mathrm{T}(\mathrm{S}_i)$ which acts trivially on $\mathrm{V}_{0,i}$, $\mathrm{T}_2(\mathrm{S}_i)$ the subgroup of $\mathrm{T}(\mathrm{S}_i)$ such that $\mathrm{T}(\mathrm{S}_i) = \mathrm{T}_1(\mathrm{S}_i) \times \mathrm{T}_2(\mathrm{S}_i)$ with $\mathrm{T}_2(\mathrm{S}_i) \subseteq \mathrm{G}(\mathrm{V}_{0,i})$. In particular,

$$\mathrm{H}(\mathrm{S}_i) = \mathrm{T}_1(\mathrm{S}_i) \times \mathrm{A}(\mathrm{S}_i) \times \mathrm{T}_2(\mathrm{S}_i) \subseteq \mathrm{GL}(\mathrm{X}_i) \times \mathrm{G}(\mathrm{V}_{0,i}). \tag{4}$$

  Similarly,

$$\mathrm{H}'(\mathrm{S}'_j) = \mathrm{T}'_1(\mathrm{S}'_j) \times \mathrm{A}'(\mathrm{S}'_j) \times \mathrm{T}'_2(\mathrm{S}'_j) \subseteq \mathrm{GL}(\mathrm{X}_j) \times \mathrm{G}(\mathrm{U}_j). \tag{5}$$

- $\mathrm{P}(\mathrm{S}_i)$ and $\mathrm{P}'(\mathrm{S}'_j)$ be the parabolic subgroups of G and G′ whose Levi parabolic $\mathrm{L}(\mathrm{S}_i)$ and $\mathrm{L}'(\mathrm{S}'_j)$ are given by:

$$\mathrm{L}(\mathrm{S}_i) = \mathrm{GL}(\mathrm{X}_i) \times \mathrm{G}(\mathrm{V}_{0,i}), \qquad \mathrm{L}'(\mathrm{S}'_i) = \mathrm{GL}(\mathrm{X}_i) \times \mathrm{G}(\mathrm{U}_i).$$

  and $\mathrm{N}(\mathrm{S}_i)$ and $\mathrm{N}'(\mathrm{S}'_i)$ be the corresponding unipotent subgroups.
- $\mathrm{W}_{0,i}$ the subspace of W defined by $\mathrm{Hom}(\mathrm{U}_i, \mathrm{V}_{0,i})$, $(\mathrm{G}(\mathrm{U}_i), \mathrm{G}(\mathrm{V}_{0,i}))$ dual pair in $\mathrm{Sp}(\mathrm{Hom}(\mathrm{U}_i, \mathrm{V}_{0,i}))$.

Remark 4.5. One can easily check that the forms on $\mathrm{V}_{0,i}$ and U_i have signature $(p-i, q-i)$ and $(r-i, s-i)$ respectively.

For every $\tilde{h} = \tilde{t}_1\tilde{a}\tilde{t}_2 \in \widetilde{\mathrm{H}}(\mathrm{S}_i)^{\mathrm{reg}}$ (using the decomposition of $\mathrm{H}(\mathrm{S}_i)$ given in Equation (4)) and $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$, we get:

(6)
$$|\det(\mathrm{Ad}(\tilde{h}) - \mathrm{Id})_{\eta(\mathrm{S}_i)}|\mathrm{Chc}_{\tilde{h}}(\Psi) =$$
$$\mathrm{C}_i \mathrm{d}_{\mathrm{S}_i}(\tilde{h})\varepsilon(\tilde{t}_1\tilde{a}) \int_{\mathrm{GL}(\mathrm{X}_i)/\mathrm{T}_1(\mathrm{S}_i)\times\mathrm{A}(\mathrm{S}_i)} \int_{\widetilde{\mathrm{G}}(\mathrm{U}_i)} \left(\varepsilon(\tilde{t}_1\tilde{a}\tilde{y})\mathrm{d}'_{\mathrm{S}_i}(g\tilde{t}_1\tilde{a}g^{-1}\tilde{y})\Psi^{\widetilde{\mathrm{K}'}}_{\widetilde{\mathrm{N}'}(\mathrm{S}_i)}(g\tilde{t}_1\tilde{a}g^{-1}\tilde{y})\right)\mathrm{Chc}_{\mathrm{W}_{0,i}}(\tilde{t}_2\tilde{y})d\tilde{y}d\overline{g}\,,$$

where C_i is a constant, $\mathrm{d}_{\mathrm{S}_i} : \widetilde{\mathrm{L}}(\mathrm{S}_i) \to \mathbb{R}$ and $\mathrm{d}'_{\mathrm{S}'_i} : \widetilde{\mathrm{L}'}(\mathrm{S}'_i) \to \mathbb{R}$ are given by

$$\mathrm{d}_{\mathrm{S}_i}(\tilde{l}) = |\det(\mathrm{Ad}(\tilde{l})_{\eta(\mathrm{S}_i)})|^{\frac{1}{2}}, \qquad \mathrm{d}'_{\mathrm{S}'_i}(\tilde{l}') = |\det(\mathrm{Ad}(\tilde{l}')_{\eta'(\mathrm{S}_i)})|^{\frac{1}{2}}, \qquad \left(\tilde{l} \in \widetilde{\mathrm{L}}(\mathrm{S}_i), \tilde{l}' \in \widetilde{\mathrm{L}'}(\mathrm{S}'_i)\right),$$

and $\Psi^{\widetilde{\mathrm{K}'}}_{\widetilde{\mathrm{N}'}(\mathrm{S}'_i)}$ is the Harish-Chandra transform of Ψ, i.e. the function on $\widetilde{\mathrm{L}'}(\mathrm{S}'_i)$ defined by:

$$\Psi^{\widetilde{\mathrm{K}'}}_{\widetilde{\mathrm{N}'}(\mathrm{S}'_i)}(\tilde{l}') = \int_{\widetilde{\mathrm{N}'}(\mathrm{S}'_i)} \int_{\widetilde{\mathrm{K}'}} \Psi(\tilde{k}\tilde{l}'\tilde{n}\tilde{k}^{-1})d\tilde{k}d\tilde{n}, \qquad \left(\tilde{l}' \in \widetilde{\mathrm{L}'}(\mathrm{S}'_i)\right).$$

5. Discrete series representations

Definition 5.1. We say that $(\Pi, (\mathscr{H}, \langle\cdot,\cdot\rangle))$ is a discrete series representation if all the functions

$$\tau_{u,v} : \mathrm{G} \ni g \to \langle \Pi(g)u, v\rangle \in \mathbb{C}$$

are in $\mathrm{L}^2(\mathrm{G})$.

Theorem 5.2. G *has discrete series if and only if* G *has a compact Cartan subgroup.*

In [7] and [6], Harish-Chandra classified the discrete series representations. More precisely, we proved the following theorem:

Theorem 5.3. *Let λ be an element of $i\mathfrak{h}^*$ such that $\lambda+\rho$ is analytic integral. Then, there exists a discrete series representation $(\Pi_\lambda, \mathscr{H}_\lambda)$ of* G *such that:*

1. *The representation Π_λ has infinitesimal character χ_λ, where χ_λ is defined in Remark 5.4,*
2. *The linear form $\nu = \lambda + \rho - 2\rho(\mathfrak{k})$ is the highest weight of the lowest* K*-type for $\Pi_{\lambda_{|\mathrm{K}}}$ and the multiplicity of the corresponding representation Π_ν in $\Pi_{\lambda_{|\mathrm{K}}}$ is one.*

The parameter λ is called the Harish-Chandra parameter of Π_λ.

For such λ, we denote by $(\Pi_\lambda, \mathscr{H}_\lambda)$ the corresponding discrete series of G and by Θ_{Π_λ} its character.

Remark 5.4. Let $\lambda \in i\mathfrak{h}^*$. We get a linear map $\lambda : \mathfrak{h}_\mathbb{C} \to \mathbb{C}$ which can be extended to a map $\lambda : \mathrm{S}(\mathfrak{h}_\mathbb{C}) \to \mathbb{C}$. Using that $\mathrm{Z}(\mathscr{U}(\mathfrak{g}_\mathbb{C})) \approx \mathrm{S}(\mathfrak{h}_\mathbb{C})^{\mathscr{W}}$ (see [11, Chapter 8.5]), we get a character of $\mathrm{Z}(\mathscr{U}(\mathfrak{g}_\mathbb{C}))$. We will denote this character by χ_λ.

Proposition 5.5.
1. *For every $z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}_\mathbb{C}))$, $z\Theta_{\Pi_\lambda} = \chi_\lambda(z)\Theta_{\Pi_\lambda}$, where χ_λ is defined in Remark 5.4.*
2. *For every $X \in \mathfrak{h}^{\mathrm{reg}}$, we get:*

$$\Theta_{\Pi_\lambda}(\exp(X)) = (-1)^{\dim(\mathrm{G}/\mathrm{K})} \sum_{w\in\mathscr{W}(\mathfrak{k})} \varepsilon(w) \frac{e^{w\lambda(X)}}{\prod\limits_{\alpha\in\Psi(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})} \left(e^{\frac{\alpha(X)}{2}} - e^{-\frac{\alpha(X)}{2}}\right)}.$$

3. $\sup\limits_{g\in\mathrm{G}^{\mathrm{reg}}} |\Delta(g)|^{\frac{1}{2}}|\Theta_{\Pi_\lambda}(g)| < \infty.$

Theorem 5.6. *Let $\Theta(= \Theta_\lambda)$ be a* G*-invariant eigendistribution satisfying the three conditions of Proposition 5.5. Then, Θ is the character of a discrete series representation of* G *of Harish-Chandra parameter λ.*

6. Results of Annagret Paul

In this section, we assume that $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p,q), \mathrm{U}(r,s))$. As explained in [14, Section 1.2], the nature of the double cover of G is isomorphic to

$$\left\{(g,\xi) \in \mathrm{G} \times \mathbb{C}^*, \xi^2 = \det{}^{r-s}(g)\right\} .$$

We first recall a result of J.S. Li.

Theorem 6.1. [12, Proposition 2.4]
Assume that $p + q \leq r + s$ and Π be a discrete series of $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$. Then, Π can be embedded as a subrepresentation of ω (and $\Pi'_1 = \Pi'$ is also a subrepresentation of ω).

From now on, we fix (p,q) and let (r,s) vary under the condition $p + q = r + s$.

Theorem 6.2. [14, Theorem 0.1] *and* [15, Theorem 2.7]

(1) For every discrete series representation Π of $\widetilde{\mathrm{U}}(p,q)$, there exists a unique (r,s) such that $p+q = r = s$ and $\theta_{r,s}(\Pi) \neq 0$.
(2) The corresponding representation $\Pi'_1 = \Pi'$ is a discrete series representation of $\widetilde{\mathrm{U}}(r,s)$.
(3) If λ is the Harish-Chandra parameter of Π

$$\lambda = \lambda_{a,b} = \left(\alpha_1, \ldots, \alpha_a, \beta_1, \ldots, \beta_{p-a}, \gamma_1, \ldots, \gamma_b, \delta_1, \ldots, \delta_{q-b}\right) ,$$

with $\alpha_i, \beta_j, \gamma_k, \delta_l \in \mathbb{Z} + \frac{1}{2}$ such that $\alpha_1 > \ldots > \alpha_a > 0 > \beta_1 > \ldots > \beta_{p-a}$ and $\gamma_1 > \ldots > \gamma_b > 0 > \delta_1 > \ldots > \delta_{q-b}$, then $(r_\Pi, s_\Pi) = (a + q - b, b + p - a)$ and the corresponding Harish-Chandra parameter $\lambda' = \lambda'_{a,b}$ of $\theta_{r_\Pi, s_\Pi}(\Pi)$ is of the form:

$$\lambda'_{a,b} = \left(\alpha_1, \ldots, \alpha_a, \delta_1, \ldots, \delta_{q-b}, \gamma_1, \ldots, \gamma_b, \beta_1, \ldots, \beta_{p-a}\right) .$$

7. Idea of the proof

Let Π be a discrete series of $\widetilde{\mathrm{U}}(p,q)$ of Harish-Chandra parameter $\lambda = \lambda_{a,b}$, $0 \leq a \leq p, 0 \leq b \leq q$. Let $\Theta'_\Pi = \mathrm{Chc}^*(\Theta_\Pi)$.

(1) According to Theorems 1.4 and 3.7, there exists a locally integrable function Θ'_Π on $\widetilde{\mathrm{G}'}$, analytic on $\widetilde{\mathrm{G}'}^{\mathrm{reg}}$, such that

$$\Theta'_\Pi(\Psi) = \int_{\widetilde{\mathrm{G}'}} \Theta'_\Pi(\tilde{g}')\Psi(\tilde{g}')d\tilde{g}', \qquad \left(\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})\right) .$$

(2) We prove that there exists $\sigma_{a,b} \in \mathscr{S}_{p+q}$ such that for every $X \in \mathfrak{h}^{\mathrm{reg}}$,

$$\Delta_{\Psi'}(X)\Theta'_\Pi(\widetilde{\exp}(X)) = \mathrm{C} \sum_{w \in \mathscr{S}_r \times \mathscr{S}_s} \varepsilon(\sigma) e^{w(\sigma_{a,b}\lambda)(X)} ,$$

where C is a real number.

(3) We prove that $\sup\limits_{\tilde{g}' \in \widetilde{\mathrm{G}'}^{\mathrm{reg}}} |\Delta_{\Psi'}(\tilde{g}')||\Theta'_\Pi(\tilde{g}')| < \infty$.

(4) Using [2, Theorem 1.4], we get $z \in \mathrm{Z}(\mathscr{U}(\mathfrak{gl}(p+q,\mathbb{C}))) = \mathrm{Z}(\mathscr{U}(\mathfrak{g}_\mathbb{C})) = \mathrm{Z}(\mathscr{U}(\mathfrak{g}'_\mathbb{C}))$, we get

$$z\Theta'_\Pi = z\mathrm{Chc}^*(\Theta_\Pi) = \mathrm{Chc}^*(z\Theta_\Pi).$$

Because Π is a discrete series, we get:

$$\mathrm{Chc}^*(z\Theta_\Pi) = \mathrm{Chc}^*(\chi_\lambda(z)\Theta_\Pi) = \chi_\lambda(z)\mathrm{Chc}^*(\Theta_\Pi) ,$$

and because $\mathrm{Spec}(\mathrm{Z}(\mathscr{U}(\mathfrak{g}'_{\mathbb{C}}))) = \mathfrak{h}'/\mathscr{S}_{r+s}$, it follows that $\chi_\lambda = \chi_{\sigma_{a,b}\lambda}$ and then,

$$z\Theta'_\Pi = \chi_{\sigma_{a,b}\lambda}(z)\Theta'_\Pi, \qquad \left(z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}'_{\mathbb{C}}))\right).$$

In particular, it follows that Θ'_Π is the character of a discrete series representation of $\widetilde{\mathrm{U}}(r,s)$ of Harish-Chandra parameter $\sigma_{a,b}\lambda$.

(5) We prove that $\Theta'_\Pi = \Theta_{\Pi'}$. Using [11, Theorem 9.20], it follows that two discrete series representation are equivalent if and only if the corresponding Harish-Chandra parameters are conjugated under the compact Weyl group. In particular, using Theorem 6.2, we prove that $\lambda'_{a,b}$ and $\sigma_{a,b}(\lambda_{a,b})$ are conjugated under $\mathscr{S}_r \times \mathscr{S}_s$. In particular,

$$\mathrm{Chc}^*(\Theta_\Pi) = \Theta_{\Pi'} = \Theta_{\Pi'_1}.$$

8. A current project

Assume that $(\mathrm{G}, \mathrm{G}')$ is such that $\mathrm{rk}(\mathrm{G}) \leq \mathrm{rk}(\mathrm{G}')$ and let $(\Pi, \mathscr{H}_\Pi)$ be a discrete series representation Π of $\widetilde{\mathrm{G}}$. As in Theorem 5.3, let ν the minimal $\widetilde{\mathrm{K}}$-type of Π and let $(\Pi_\nu, \mathscr{H}_\nu)$ be the corresponding irreducible $\widetilde{\mathrm{K}}$-modules.

Theorem 8.1. *Let d_Π be the formal degree of Π (see [11, Chapter 9.3]). The operator $\omega(d_\Pi\overline{\Theta_\Pi})$ is well-defined and its a projection operator onto the Π-isotypic component $\mathscr{H}(\Pi)$, where*

$$\mathscr{H}(\Pi) = \overline{\left\{T(\mathscr{H}_\Pi), T \in \mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathscr{H}_\Pi, \mathscr{H})\right\}}.$$

Proof. See [13, Section 7]. □

We denote by $\mathscr{P}_\Pi : \mathscr{H} \to \mathscr{H}(\Pi)$ the previous operator. As a $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}'}$-module, we have $\mathscr{H}(\Pi) = \Pi \otimes \Pi'$. As a $\widetilde{\mathrm{K}}$-module, we get $\Pi_{|_{\widetilde{\mathrm{K}}}} = \Pi_\nu \oplus \left(\sum\limits_{\xi \neq \nu} m_\xi \Pi_\xi\right)$, where Π_ξ is the irreducible $\widetilde{\mathrm{K}}$-module of highest weight ξ. In particular, as a $\widetilde{\mathrm{K}} \times \widetilde{\mathrm{G}'}$, the Π-isotypic component $\mathscr{H}(\Pi)$ can be decomposed as

$$\mathscr{H}(\Pi) = \Pi_\nu \otimes \Pi' \oplus \left(\sum_{\xi \neq \nu} \Pi_\xi \otimes \Pi'\right).$$

We denote by $\mathscr{H}(\Pi)(\nu)$ the ν-isotypic component in $\mathscr{H}(\Pi)$ and by $\mathscr{P}^\Pi_\nu : \mathscr{H}(\Pi) \to \mathscr{H}(\Pi)(\nu)$ the corresponding projection operator. In particular, $\mathscr{P}^\Pi_\nu = \Pi(d_\nu\overline{\Theta_\nu})$, where Θ_ν is the character of Π_ν and d_ν the dimension of $\mathscr{H}_\nu$. In particular, as a $\widetilde{\mathrm{K}} \times \widetilde{\mathrm{G}'}$-module, $\mathscr{H}(\Pi)(\nu) = \Pi_\nu \otimes \Pi'$.
Let $\mathscr{P} = \mathscr{P}^\Pi_\nu \circ \mathscr{P}_\Pi : \mathscr{H} \to \mathscr{H}(\Pi) \to \mathscr{H}(\Pi)(\nu)$. One can easily check that:

$$\mathscr{P} = \Pi(d_\nu\overline{\Theta_\nu}) \circ \omega(d_\Pi\overline{\Theta_\Pi}) = \omega(d_\nu\overline{\Theta_\nu}) \circ \omega(d_\Pi\overline{\Theta_\Pi}).$$

In particular, for every $\Psi \in \mathscr{C}^\infty_c(\widetilde{\mathrm{G}'})$, we get:

$$\begin{aligned}
\Theta_{\Pi'}(\Psi) &= \mathrm{tr}(\Pi'(\Psi)) = \frac{1}{d_\nu}\mathrm{tr}(\mathrm{Id}_{\mathscr{H}_\nu} \otimes \Pi'(\Psi)) = \frac{1}{d_\nu}\mathrm{tr}(\mathscr{P} \circ \omega(\Psi)) \\
&= d_\Pi \mathrm{tr} \int_{\widetilde{\mathrm{K}}} \int_{\widetilde{\mathrm{G}}} \int_{\widetilde{\mathrm{G}'}} \overline{\Theta_\nu(\tilde{k})}\overline{\Theta_\Pi(\tilde{g})}\omega(\tilde{k}\tilde{g}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}'d\tilde{g}d\tilde{k} \\
&= \sum_{i=1}^n \frac{1}{|\mathscr{W}(\mathrm{H}_i)|} \int_{\widetilde{\mathrm{H}}_i} \overline{\Theta_\Pi(\tilde{h}_i)} \left|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}_i)^{-1})_{\mathfrak{g}/\mathfrak{h}_i}\right| \left(\int_{\widetilde{\mathrm{K}}} \int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{H}}_i} \int_{\widetilde{\mathrm{G}'}} \overline{\Theta_\nu(\tilde{k})}\omega(\tilde{k}\tilde{g}\tilde{h}_i\tilde{g}^{-1}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}'d\tilde{g}\widetilde{\mathrm{H}}_i d\tilde{k}\right) d\tilde{h}_i,
\end{aligned}$$

where $H_1, \dots, H_n$ is a maximal set of Cartan subgroups of G.
In particular, if $(G, G') = (U(p,q), U(r,s))$, $p+q = r+s$, we get the following equality:

$$\sum_{i=0}^{p} \frac{1}{|\mathscr{W}(H_i)|} \int_{\widetilde{H}_i} \overline{\Theta_\Pi(\tilde{h}_i)} \left|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}_i)^{-1})_{\mathfrak{g}/\mathfrak{h}_i}\right| \mathrm{Chc}_{\tilde{h}_i}(\Psi) d\tilde{h}_i =$$

$$\sum_{i=0}^{p} \frac{1}{|\mathscr{W}(H_i)|} \int_{\widetilde{H}_i} \overline{\Theta_\Pi(\tilde{h}_i)} \left|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}_i)^{-1})_{\mathfrak{g}/\mathfrak{h}_i}\right| \left(\int_{\widetilde{K}} \int_{\widetilde{G}/\widetilde{H}_i} \int_{\widetilde{G}'} \overline{\Theta_\nu(\tilde{k})} \omega(\tilde{k}\tilde{g}\tilde{h}_i\tilde{g}^{-1}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}' d\tilde{g}\widetilde{H}_i d\tilde{k}\right) d\tilde{h}_i \,.$$

Appendix A. Construction of Chc*

For every $\Psi \in \mathscr{C}_c^\infty(G)$, we define the orbital integral $J_G(\Psi)$ of G as follow: let $\gamma \in G^{reg}$ and $H(\gamma)$ be the unique Cartan subgroup of Car(G) containing γ, the value of $J_G(\Psi)$ at γ is given by

$$J_G(\Psi)(\gamma) = |\det(\mathrm{Id} - \mathrm{Ad}(\gamma)^{-1})_{\mathfrak{g}/\mathfrak{h}(\gamma)}|^{\frac{1}{2}} \int_{G/H(\gamma)} \Psi(g\gamma g^{-1}) \overline{dg} \,.$$

One can easily check that $J_G \in \mathscr{C}^\infty(G^{reg})^G$.
The construction of Chc* is mainly based on Bouaziz's paper [3], who gave a parametrization of the set of orbital integrals inside $\mathscr{C}^\infty(G^{reg})^G$.

Definition A.1. (1) A subset Ω of G is said completely invariant if for any compact subset C of Ω, then

$$\overline{\{gCg^{-1}, g \in G\}} \subset \Omega \,.$$

(2) Let $\mathscr{U}$ be a subspace of G. We say that $\mathscr{U}$ is compact modulo G if $\mathscr{U}$ is closed, G-invariant and such that $\mathscr{U} \cap H$ is compact for every $H \in \mathrm{Car}(G)$.

Lemma A.2. *Let $\mathscr{U}$ be a completely* G*-invariant subspace of* $\mathfrak{g}$*. There exists a sequences of open and completely* G*-invariant subspaces $\{\mathscr{U}_n\}_{n\in\mathbb{Z}^+}$ covering $\mathscr{U}$ such that for every $n \in \mathbb{Z}^+$, $\overline{\mathscr{U}_n}$ is compact modulo* G *and included in $\mathscr{U}_{n+1}$.*

For $\mathfrak{h} \in \mathrm{Car}(\mathfrak{g})$, we denote $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the set of roots of $(\mathfrak{g}, \mathfrak{h})$ and by $\Delta_I = \Delta_I(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the set of imaginary roots, i.e. $\Delta_I = \left\{\alpha \in \Delta, \alpha_{|\mathfrak{h}} \in i\mathbb{R}\right\}$. Let Ψ_I be a set of positive roots of Δ_I. For every $\alpha \in \Psi_I$, we denote by $\mathfrak{s}_\alpha = \mathfrak{g}_{\alpha,\mathbb{C}} \oplus [\mathfrak{g}_{\alpha,\mathbb{C}}, \mathfrak{g}_{-\alpha,\mathbb{C}}] \oplus \mathfrak{g}_{-\alpha,\mathbb{C}}$ the three dimensional Lie subalgebras of $\mathfrak{g}_\mathbb{C}$.

Definition A.3. We say that α is compact (resp. non-compact) if $\mathfrak{s}_\alpha \cap \mathfrak{g}$ is isomorphic to $\mathfrak{su}(2)$ (resp. $\mathfrak{sl}(2,\mathbb{R})$).
We denote by H_I the subset of H defined by $H_I = \{h \in H, h^\alpha \neq 1, \alpha \in \Delta_I \text{ non compact}\}$.

Let $\mathscr{U}$ be an open and completely G-invariant subspace of G. We denote by $\mathcal{I}(\mathscr{U})$ the subspace of functions $\Psi \in \mathscr{C}^\infty(\mathscr{U}^{reg})^G$ satisfying the following four conditions:

(1) If $H \in \mathrm{Car}(G)$ and every compact K of $H \cap \mathscr{U}$,

$$\sup_{h \in K \cap \mathscr{U}^{reg}} |\partial(u)\Psi_H(h)| < \infty \,,$$

for every $u \in S(\mathfrak{h}_\mathbb{C})$, where Ψ_H is the restriction of Ψ to $H \cap \mathscr{U}^{reg}$ and $\partial(u)$ is the differential operator on H corresponding to u.

(2) If $H \in \mathrm{Car}(G)$ and every system of positive roots Ψ_I of Δ_I, the function $b_{\Psi_I}\Psi_H$ can be extended as a smooth function on $H_I \cap \mathscr{U}$, where b_{Ψ_I} is the function on H^{reg} given by

$$b_{\Psi_I}(h) = \prod_{\alpha \in \Psi_I} \frac{1 - h^{-\alpha}}{|1 - h^{-\alpha}|}, \qquad (h \in H^{reg}) \,.$$

(3) For every $\mathrm{H} \in \mathrm{Car}(\mathrm{G})$, $\mathrm{supp}_{\mathrm{H}}(\Psi_{\mathrm{H}})$ is a compact included in $\mathrm{H} \cap \mathscr{U}$, where

$$\mathrm{supp}_{\mathrm{H}}(\Psi_{\mathrm{H}}) = \overline{\{h \in \mathrm{H} \cap \mathscr{U}^{\mathrm{reg}}, \Psi_{\mathrm{H}}(h) \neq 0\}}^{\mathrm{H}} .$$

(4) For every $(s, \mathrm{H}, \Psi, \mathrm{H}', \Psi', \alpha, c_\alpha)$ as in [3, Page 580], we get for every $u \in \mathrm{S}(\mathfrak{h}_{\mathbb{C}})$ that

$$\begin{aligned}\lim_{t\to 0^+} \partial(\tau_\Psi(u))\mathrm{b}_\Psi\Psi_{\mathrm{H}}(s\exp(it\mathrm{H}_\alpha)) - \lim_{t\to 0^-} \partial(\tau_\Psi(u))\mathrm{b}_\Psi\Psi_{\mathrm{H}}(s\exp(it\mathrm{H}_\alpha)) \\ = i\mathrm{d}(s)\partial(\tau_{\Psi'}(c_\alpha u))\Psi_{\mathrm{H}'}\mathrm{b}_{\Psi'}(s) ,\end{aligned}$$

where $d(s) \in \{1, 2\}$.

We now define a topology on $\mathcal{I}(\mathscr{U})$. If $\mathscr{L} \subseteq \mathscr{U}$ is compact modulo G, we define a topology on $\mathcal{I}(\mathscr{L})$ by using the family of semi-norms

$$\nu_{\mathrm{H},u}(\Psi) = \sup_{h\in\mathrm{H}} |\partial(u)\Psi_{\mathrm{H}}(h)|, \qquad (\mathrm{H} \in \mathrm{Car}(\mathrm{G}), u \in \mathrm{S}(\mathfrak{h}_{\mathbb{C}}), \Psi \in \mathcal{I}(\mathscr{L})) .$$

The space $\mathcal{I}(\mathscr{L})$ is a Frechet space. If $\mathscr{L}, \mathscr{L}'$ are two compact modulo G subsets of $\mathscr{U}$ such that $\mathscr{L} \subseteq \mathscr{L}'$, the space $\mathcal{I}(\mathscr{L})$ is naturally embedded into $\mathcal{I}(\mathscr{L}')$. In particular, by considering the sequence $\{\mathrm{W}_n\}_{n\in\mathbb{Z}^+}$ of Lemma A.2, we define a topology on $\mathcal{I}(\mathscr{U})$ by taking the inductive limit of the sequence $\mathcal{I}(\mathrm{W}_n)_{n\in\mathbb{Z}^+}$, i.e.

$$\mathcal{I}(\mathscr{U}) = \lim_{\overleftarrow{n}} \mathcal{I}(\mathrm{W}_n) .$$

Theorem A.4. *For every $\Psi \in \mathscr{C}_c^\infty(\mathscr{U})$, $\mathrm{J}_{\mathscr{U}}(\Psi) \in \mathcal{I}(\mathscr{U})$ and the corresponding map*

$$\mathrm{J}_{\mathscr{U}} : \mathscr{C}_c^\infty(\mathscr{U}) \to \mathcal{I}(\mathscr{U})$$

is continuous and surjective.

By the continuity of $\mathrm{J}_{\mathscr{U}}$, we get that the map $\mathrm{J}^t_{\mathscr{U}} : \mathcal{I}(\mathscr{U})^* \to \mathscr{D}'(\mathscr{U})$ given by

$$\mathrm{J}^t_{\mathscr{U}}(\mathrm{T})(\Psi) = \mathrm{T}(\mathrm{J}_{\mathscr{U}}\Psi), \qquad (\mathrm{T} \in \mathcal{I}(\mathscr{U})^*, \Psi \in \mathscr{C}_c^\infty(\mathscr{U})) ,$$

where $\mathcal{I}(\mathscr{U})^*$ is the set of continuous linear forms on $\mathcal{I}(\mathscr{U})$. One can easily check that for every $\mathrm{T} \in \mathcal{I}(\mathscr{U})^*$, the distribution $\mathrm{J}^t_{\mathscr{U}}(\mathrm{T})$ is G-invariant.

Theorem A.5. *The map $\mathrm{J}^t_{\mathscr{U}} : \mathcal{I}(\mathscr{U})^* \to \mathscr{D}'(\mathscr{U})^{\mathrm{G}}$ is bijective.*

We are able to construct Chc^*. Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair in Sp(W). We have a map (Equation (2)):

$$\widetilde{\mathrm{Chc}} : \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'}) \to \mathscr{C}^\infty(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}} .$$

Theorem A.6. *For every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$, $\widetilde{\mathrm{Chc}}(\Psi) \in \mathcal{I}(\widetilde{\mathrm{G}})$. Moreover, for every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$ such that $\mathrm{J}_{\widetilde{\mathrm{G}'}}(\Psi) = 0$, then $\widetilde{\mathrm{Chc}}(\Psi) = 0$. In particular, we get a map*

$$\widetilde{\mathrm{Chc}} : \mathcal{I}(\widetilde{\mathrm{G}'}) \to \mathcal{I}(\widetilde{\mathrm{G}}) \tag{7}$$

and the map is continuous.

By dualizing the map (7), we get a map:

$$\widetilde{\mathrm{Chc}}^t : \mathcal{I}(\widetilde{\mathrm{G}})^* \to \mathcal{I}(\widetilde{\mathrm{G}'})^* ,$$

and by using the isomorphisms $\mathrm{J}^t_{\widetilde{\mathrm{G}}} : \mathcal{I}(\widetilde{\mathrm{G}})^* \to \mathscr{D}'(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}}$ and $\mathrm{J}^t_{\widetilde{\mathrm{G}'}} : \mathcal{I}(\widetilde{\mathrm{G}'})^* \to \mathscr{D}'(\widetilde{\mathrm{G}'})^{\widetilde{\mathrm{G}'}}$, we get a map $\mathrm{Chc}^* : \mathscr{D}'(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}} \to \mathscr{D}'(\widetilde{\mathrm{G}'})^{\widetilde{\mathrm{G}'}}$ given by

$$\mathrm{Chc}^* = \mathrm{J}^t_{\widetilde{\mathrm{G}'}} \circ \widetilde{\mathrm{Chc}}^t \circ (\mathrm{J}^t_{\widetilde{\mathrm{G}}})^{-1} .$$

Appendix B. Cartan subgroups of unitary groups

We assume that $p \leq q$.

- Number of non-conjugated Cartan subgroups of $\mathrm{G} = \mathrm{U}(p,q)$: $p+1$.
- $\mathrm{K} = \mathrm{U}(p) \times \mathrm{U}(q)$ maximal compact subgroup of G,
- H (diagonal) compact Cartan subgroup of K,
- $\mathfrak{h}$, $\mathfrak{k}$ and $\mathfrak{g}$ the Lie algebras of H, K and G respectively, $\mathfrak{h}_{\mathbb{C}}$, $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{C}}$ their complexifications.
- $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \left\{\pm(e_i - e_j), 1 \leq i < j \leq p+q\right\}$ be the set of roots,
- $\Delta_c := \Delta_c(\mathfrak{k}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \left\{\pm(e_i - e_j), 1 \leq i < j \leq p\right\} \cup \left\{\pm(e_i - e_j), p+1 \leq i < j \leq p+q\right\}$ the set of compact roots,
- $\Delta_n = \Delta \setminus \Delta_c = \left\{\pm(e_i - e_j), 1 \leq i \leq p, p+1 \leq j \leq p+q\right\}$ the set of non-compact roots.
- Ψ set of positive roots of Δ, Ψ_c and Ψ_n the subsets of Ψ given by $\Psi_c = \Delta_c \cap \Psi$ and $\Psi_n = \Delta_n \cap \Psi$. In particular,

$$\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\mathbb{C},\alpha} \,,$$

  where $\mathfrak{g}_{\mathbb{C},\alpha} = \{X \in \mathfrak{g}_{\mathbb{C}}, [H,X] = \alpha(H)X, H \in \mathfrak{h}_{\mathbb{C}}\}$.

Notation B.1. For every $\alpha \in \Delta$, we fix $X_\alpha \in \mathfrak{g}_{\mathbb{C},\alpha}, Y_\alpha \in \mathfrak{g}_{\mathbb{C},-\alpha}$ and $H_\alpha \in i\mathfrak{h}$ such that:

$$[H_\alpha, X_\alpha] = 2X_\alpha, \qquad [H_\alpha, Y_\alpha] = -2Y_\alpha, \qquad [X_\alpha, Y_\alpha] = H_\alpha, \qquad \overline{H_\alpha} = -H_\alpha = H_{-\alpha}\,,$$

and such that $\overline{X_\alpha} = -Y_\alpha$ if $\alpha \in \Delta_c$ and $\overline{X_\alpha} = Y_\alpha$ if $\alpha \in \Delta_n$.

Definition B.2. We say that $\alpha, \beta \in \Delta$ are strongly orthogonal if $\alpha \neq \pm\beta$ and $\alpha \pm \beta \notin \Delta$. We denote by Ψ_n^{st} a maximal family of strongly orthogonal roots of Ψ_n (i.e. a subset of Ψ_n such that every pairs $\alpha, \beta \in \Psi_n^{\mathrm{st}}$ are strongly orthogonal).

In particular, for G,

$$\Psi_n^{\mathrm{st}} = \left\{e_1 - e_{p+1}, \ldots, e_p - e_{2p}\right\} .$$

For every $\alpha \in \Psi_n^{\mathrm{st}}$, we denote by $c(\alpha)$ the element of $\mathrm{GL}(p+q, \mathbb{C})$ given by:

$$c(\alpha) = \exp\left(\frac{i\pi}{4}(Y_\alpha - X_\alpha)\right) .$$

For every subset S of Ψ_n^{st}, we denote by $c(\mathrm{S})$ the element of $\mathrm{GL}(p+q, \mathbb{C})$ defined by

$$c(\mathrm{S}) = \prod_{\alpha \in \mathrm{S}} c(\alpha)\,,$$

and let

$$\mathfrak{h}(\mathrm{S}) = \mathfrak{g} \cap \mathrm{Ad}(c(\mathrm{S}))(\mathfrak{h}_{\mathbb{C}})\,.$$

We denote by H(S) the analytic subgroup of G whose Lie algebra is $\mathfrak{h}(\mathrm{S})$. Then, H(S) is a Cartan subgroup of G and one can prove that all the Cartan subgroups are of this form (up to conjugation).
For every $\mathrm{S} \subseteq \Psi_n^{\mathrm{st}}$, we will denote by $\mathrm{H_S}$ the subgroup of $\mathrm{H}_{\mathbb{C}}$ given by:

$$\mathrm{H_S} = c(\mathrm{S})^{-1}\mathrm{H}(\mathrm{S})c(\mathrm{S})\,.$$

where $\mathrm{H}_{\mathbb{C}} = \left\{\mathrm{diag}(\lambda_1, \ldots, \lambda_{p+q}), \lambda_i \in \mathbb{C}\right\}$.

Example B.3. Let $\mathrm{G} = \mathrm{U}(1,1)$ and $\mathrm{H} = \{\lambda = \mathrm{diag}(\lambda_1, \lambda_2), \lambda_1, \lambda_2 \in \mathrm{U}(1)\}$. In this case, $\Psi = \Psi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) = \{\alpha = e_1 - e_2\}$, $\mathrm{S}_1 = \{e_1 - e_2\}$,

$$c(\mathrm{S}_1) = \exp\left(\frac{i\pi}{4}(\mathrm{E}_{2,1} - \mathrm{E}_{1,2})\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

and

$$\mathfrak{h}(\mathrm{S}_1) = \mathrm{Ad}(c(\mathrm{S}_1))(\mathfrak{h}_{\mathbb{C}}) \cap \mathfrak{g} = \left\{ \mathrm{A} = \begin{pmatrix} i\theta & X \\ X & i\theta \end{pmatrix}, X, \theta \in \mathbb{R} \right\}.$$

Similarly,

$$\mathrm{H}(\mathrm{S}_1) = \left\{ \mathrm{A} = \begin{pmatrix} e^{i\theta}\mathrm{ch}(\mathrm{X}) & \mathrm{sh}(\mathrm{X}) \\ \mathrm{sh}(\mathrm{X}) & e^{i\theta}\mathrm{ch}(X) \end{pmatrix}, \mathrm{X}, \theta \in \mathbb{R} \right\}, \qquad \mathrm{H}_{\mathrm{S}_1} = \left\{ \mathrm{A} = \begin{pmatrix} e^{i\theta - \mathrm{X}} & 0 \\ 0 & e^{i\theta + \mathrm{X}} \end{pmatrix}, X, \theta \in \mathbb{R} \right\}.$$

Remark B.4. Two Cartan subalgebras $\mathfrak{h}(\mathrm{S}_1)$ and $\mathfrak{h}(\mathrm{S}_2)$, with $\mathrm{S}_1, \mathrm{S}_2 \subseteq \Psi_n^{\mathrm{st}}$, are conjugate if and only if there exists an element of $\sigma \in \mathscr{W}$ sending $\mathrm{S}_1 \cup (-\mathrm{S}_1)$ onto $\mathrm{S}_2 \cup (-\mathrm{S}_2)$.

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