FROM SCHUR-WEYL DUALITY TO PIN-DUALITY

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Abstract. Here are some notes of a talk I gave in Oklahoma in May 2019 when I was visiting Tomasz Przebinda. We first start by recalling some basic stuffs concerning the Schur-Weyl duality and some direct consequences of this result. After that, we prove the Pin-duality for a dual pair of general linear groups $(\widetilde{\mathrm{GL}}(n,\mathbb{K}), \widetilde{\mathrm{GL}}(m,\mathbb{K})) \subseteq \mathrm{Pin}(dnm, dnm, \mathbb{R})$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $d = \dim_{\mathbb{R}}(\mathbb{K})$.

Contents

1. Motivations

Here, we recall the main ideas of [7] using the notations of [2]. Let $\mathrm{G} = \mathrm{GL}(\mathrm{V}), \mathrm{Sp}(\mathrm{V})$ or $\mathrm{O}(\mathrm{V})$, where V is a finite dimensional vector space over $\mathbb{C}$. We consider the spaces $\mathrm{U}_{\overline{0}}$ and $\mathrm{U}_{\overline{1}}$ defined by

$$\mathrm{U}_{\overline{0}} = \mathrm{V}^{\oplus m_1} \oplus \mathrm{V}^{*\oplus m_2} \qquad \mathrm{U}_{\overline{1}} = \mathrm{V}^{\oplus n_1} \oplus \mathrm{V}^{*\oplus n_2},$$

and let $\mathrm{U} = \mathrm{U}_{\overline{0}} \oplus \mathrm{U}_{\overline{1}}$ be the corresponding $\mathbb{Z}_2$-graded vector space. We denote by SS(U) the supersymmetric algebra corresponding to U, i.e.

$$\mathrm{SS}(\mathrm{U}) = \mathrm{T}(\mathrm{U})/\mathscr{T},$$

where $\mathscr{T}$ is the ideal of T(U) generated by the elements $x \otimes y - (-1)^{\alpha\beta} y \otimes x$, $x \in \mathrm{U}_\alpha$, $y \in \mathrm{U}_\beta$.

Remark 1.1. As a complex vector space, we have $\mathrm{SS}(\mathrm{U}) = \mathrm{S}(\mathrm{U}_{\overline{0}}) \otimes \Lambda(\mathrm{U}_{\overline{1}})$.

The space SS(U) has a natural $\mathbb{Z} \times \mathbb{Z}_2$-gradation. We now consider the space End(SS(U)) endowed with its natural structure of Lie superalgebra.
We will now recall the construction of the Weyl-Clifford algebra: it's a subalgebra of End(SS(U)) generated by multiplication and derivation operators on $\widetilde{\mathrm{U}} = \mathrm{U} \oplus \mathrm{U}^*$.

Remark 1.2. The space U^* has a natural $\mathbb{Z}_2$-structure coming from the one on U. In particular, we have $\widetilde{\mathrm{U}} = \widetilde{\mathrm{U}}_{\overline{0}} \oplus \widetilde{\mathrm{U}}_{\overline{1}}$, where $\widetilde{\mathrm{U}}_\alpha = \mathrm{U}_\alpha \oplus \mathrm{U}^*_\alpha$, $\alpha \in \mathbb{Z}_2$.

For all $u \in U_{\bar{0}}$, we denote by M_u the linear operator on SS(U) of degree $(1, \bar{0})$ given by:

$$M_u : SS(U) \to SS(U), \qquad M_u(v) = uv.$$

Similarly, we define an operator M_u of degree $(1, \bar{1})$ for every $u \in U_{\bar{1}}$.
For every element $u^* \in U^*_{\bar{0}}$, we define the operator D_{u^*} on SS(U) of degree $(-1, \bar{0})$ given by:

$$D_{u^*}(u_1 \dots u_n v_1 \dots v_m) = \sum_{k=1}^{n} u^*(u_k) u_1 \dots \hat{u}_k \dots u_n v_1 \dots v_m.$$

Similarly, for all $v^* \in U^*_{\bar{1}}$, we define the operator D_{v^*} of degree $(-1, \bar{1})$ given by:

$$D_{v^*}(u_1 \dots u_n v_1 \dots v_m) = \sum_{k=1}^{m} v^*(v_k) u_1 \dots u_n v_1 \dots \hat{v}_k \dots v_m.$$

We denote by $\iota : \widetilde{U} \to \mathrm{End}(SS(U))$ the map defined by

$$\iota(u) = M_u \qquad \iota(u^*) = D_{u^*} \qquad (u \in U, u^* \in U^*).$$

Remark 1.3. (1) Let b_0 be the natural symplectic form on $\widetilde{U}_{\bar{0}}$ defined by

$$b_0(v_0 + v_0^*, w_0 + w_0^*) = v_0^*(w_0) - w_0^*(v_0), \qquad \left(v_0, w_0 \in U_{\bar{0}}, v_0^*, w_0^* \in U^*_{\bar{0}}\right).$$

Note that the spaces $U_{\bar{0}}$ and $U_{\bar{1}}$ are both isotropic for b_0. Similarly, let b_1 be the non-degenerate symmetric form on $\widetilde{U}_{\bar{1}}$ given by

$$b_1(v_1 + v_1^*, w_1 + w_1^*) = v_1^*(w_1) + w_1^*(v_1), \qquad \left(v_1, w_1 \in U_{\bar{1}}, v_1^*, w_1^* \in U^*_{\bar{1}}\right).$$

and let b be the even form on $\widetilde{U}$ given by $b = b_0 \oplus b_1$ (i.e. $b(U_{\bar{0}}, U_{\bar{1}}) = \{0\}$).
Then, we get:

$$[\iota(\widetilde{u}), \iota(\widetilde{v})] = b(\widetilde{u}, \widetilde{v}).1, \qquad \left(\widetilde{u}, \widetilde{v} \in \widetilde{U}\right).$$

(2) We denote by WC(U) the subspace of End(SS(U)) generated by the elements $\iota(\widetilde{u}), \widetilde{u} \in \widetilde{U}$. We have the following property:

$$[WC(U), WC(U)] \subseteq WC(U).$$

We identify $S^2(\iota(\widetilde{U}_{\bar{0}}))$ with the elements of the form $\iota(\widetilde{a})\iota(\widetilde{b}) + \iota(\widetilde{b})\iota(\widetilde{a}), \widetilde{a}, \widetilde{b} \in \widetilde{U}_{\bar{0}}$, and $\Lambda^2(\iota(\widetilde{U}_{\bar{1}}))$ with the elements of the form $\iota(\widetilde{a})\iota(\widetilde{b}) - \iota(\widetilde{b})\iota(\widetilde{a}), \widetilde{a}, \widetilde{b} \in \widetilde{U}_{\bar{1}}$.

Notation 1.4. We denote by $\mathscr{G}_{\bar{0}} = S^2(\iota(\widetilde{U}_{\bar{0}})) \oplus \Lambda^2(\iota(\widetilde{U}_{\bar{1}}))$, $\mathscr{G}_{\bar{1}} = \iota(\widetilde{U}_{\bar{0}}) \otimes \iota(\widetilde{U}_{\bar{1}})$ and $\mathscr{G} = \mathscr{G}_{\bar{0}} \oplus \mathscr{G}_{\bar{1}}$.

Theorem 1.5. *(1) We have $[\mathscr{G}, \mathscr{G}] \subseteq \mathscr{G}$ (more precisely, $[\mathscr{G}_{\bar{0}}, \mathscr{G}_{\bar{0}}] \subseteq \mathscr{G}_{\bar{0}}$, $[\mathscr{G}_{\bar{0}}, \mathscr{G}_{\bar{1}}] \subseteq \mathscr{G}_{\bar{1}}$ and $[\mathscr{G}_{\bar{1}}, \mathscr{G}_{\bar{1}}] \subseteq \mathscr{G}_{\bar{0}}$),*
(2) $\mathscr{G} \approx \mathfrak{spo}(\widetilde{U}_{\bar{0}}, \widetilde{U}_{\bar{1}})$.

Proof. See [2, Proposition 5.5]. □

We have the different embeddings

$$G \to G \times G \to GL(U_{\bar{0}}) \times GL(U_{\bar{1}}) \to Sp(\widetilde{U}_{\bar{0}}) \times O(\widetilde{U}_{\bar{1}}),$$

and a similar embedding for the corresponding Lie algebras:

$$\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{gl}(U_{\bar{0}}) \oplus \mathfrak{gl}(U_{\bar{1}}) \to \mathfrak{sp}(\widetilde{U}_{\bar{0}}) \oplus \mathfrak{so}(\widetilde{U}_{\bar{1}}).$$

Using the identification $\mathfrak{sp}(\widetilde{\mathrm{U}}_{\bar{0}})\oplus\mathfrak{so}(\widetilde{\mathrm{U}}_{\bar{1}}) \approx \mathscr{G}_{\bar{0}}$, we denote by Γ the image of $\mathfrak{g}$ in $\mathscr{G}$ and by Γ' the centralizer of Γ in $\mathscr{G}$.

Theorem 1.6. *(1)* (Γ, Γ') *is a dual pair in* $\mathscr{G}$*,*

(2) We have the following decomposition:

$$\mathrm{SS}(\mathrm{U}) = \bigoplus_{(\mathrm{V}_\lambda,\lambda)\in\hat{\mathrm{G}}_\pi} \lambda \otimes \theta(\lambda)\,,$$

where $\hat{\mathrm{G}}_\pi$ *is the set of equivalences classes of finite dimensional irreducible representations of* G *such that* $\mathrm{Hom}_{\mathrm{G}}(\mathrm{V}_\lambda, \mathrm{SS}(\mathrm{U})) \neq \{0\}$ *and* $\theta(\lambda)$ *is an irreducible* Γ'*-module. Here, the action of* Γ' *is the one coming from the following embeddings:*

$$\Gamma' \subseteq \mathscr{G} \subseteq \mathrm{WC}(\mathrm{U}) \subseteq \mathrm{End}(\mathrm{SS}(\mathrm{U}))\,,$$

i.e. the action of Γ' *is just the restriction to* Γ' *of the natural action of* End(SS(U)) *on* SS(U).

2. Schur-Weyl duality

In this section, V will denote a finite-dimensional vector space over $\mathbb{C}$ and $\mathrm{G} = \mathrm{GL}(\mathrm{V})$ the corresponding (complex) general linear group. We denote by Π_1 the natural action of G on V and by Π_d the corresponding action on the space $\mathrm{V}^{\otimes d}$ given by:

$$\Pi_d(g)(v_1 \otimes \ldots \otimes v_d) = \Pi_1(g)(v_1) \otimes \ldots \otimes \Pi_1(g)(v_d)\,, \qquad (g \in \mathrm{GL}(\mathrm{V}), v_1, \ldots, v_d \in \mathrm{V})\;.$$

If $d \geq 2$, the representation Π_d is not irreducible. We have an explicit description of the decomposition of $\mathrm{V}^{\otimes d}$ as a direct sum of irreducible G-submodules.
On the space $\mathrm{V}^{\otimes d}$, we have an action of the symmetric group $\mathscr{S}_d$, denoted by α_d, given by

$$\alpha_d(\sigma)(v_1 \otimes \ldots \otimes v_d) = v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(d)}\,, \qquad (\sigma \in \mathscr{S}_d, v_1, \ldots, v_d \in \mathrm{V})\;.$$

Clearly, the actions Π_d and α_d commute. We have the following lemma.

Lemma 2.1. *Let* A(G) *be the subalgebra of* $\mathrm{End}(\mathrm{V}^{\otimes d})$ *generated by the operators* $\Pi_d(g), g \in \mathrm{G}$*. Then,*

$$\mathrm{Comm}_{\mathrm{End}(\mathrm{V}^{\otimes d})}\mathrm{A}(\mathrm{G}) = \mathrm{A}(\mathscr{S}_d)\,,$$

where $\mathrm{A}(\mathscr{S}_d)$ *be the subalgebra of* $\mathrm{End}(\mathrm{V}^{\otimes d})$ *generated by the operators* $\alpha_d(\sigma), \sigma \in \mathscr{S}_d$.

Proof. See [3, Section 4.2.4].

□

According to the double commutant theorem (see Appendix A), Lemma (2.1) implies that

$$\Pi_d = \bigoplus_{\lambda_d\in\mathrm{G}(\Pi_d)} \lambda_d \otimes \theta(\lambda_d)\,, \tag{1}$$

where $\mathrm{G}(\Pi_d)$ is the the set of finite dimensional irreducible representations of G such that $\mathrm{Hom}_{\mathrm{G}}(\lambda_d, \Pi_d) \neq \{0\}$ and $\theta(\lambda_d)$ is an irreducible representation of the symmetric group $\mathscr{S}_d$.
Let's be more precise concerning the representations appearing in Equation (1). An irreducible representation of the symmetric group is parameterized by a unique element of $\mathscr{C}^+$, where $\mathscr{C}^+$ is the set of all decreasing sequences of non-negatives integers with only finitely many non-zero terms. Instead of using sequences, it's common to use Young diagrams to parameterize such representations.

It makes also sense to parametrise the finite-dimensional irreducible representations of G by using those diagrams. Indeed, we know that such representations are parametrised by a "dominant" character of the diagonal torus T of G, where

$$\mathrm{T} = \{\mathrm{diag}(t_1, \ldots, t_n),\ t_1, \ldots, t_n \in \mathbb{C}^*\}\ .$$

All the characters of T have the form:

$$\Psi_\mu(\mathrm{diag}(t_1, \ldots, t_n)) = \prod_{k=1}^{n} t_k^{\mu_k}\ ,$$

where all the μ_k are integers. To a character Ψ_μ of T, we associate the partition $(\mu_1, \ldots, \mu_n)$ and then the diagram D_μ.

Theorem 2.2 (Schur-Weyl)**.** *We have the following decomposition:*

$$\mathrm{V}^{\otimes d} = \bigoplus_{\mathrm{D}} \rho_{\mathrm{V}}^{\mathrm{D}} \otimes \sigma^{\mathrm{D}}\ ,$$

where D *runs over the diagrams of size d and depth at most* $\dim_{\mathbb{C}}(\mathrm{V})$.

Proof. See [3, Section 4.2.4] for more details. □

We now deduce other results from this Schur-Weyl duality. Let U, V be two (finite dimensional) complex vector spaces and let $\mathrm{S}(\mathrm{U} \otimes \mathrm{V})$ be the symmetric algebra of the space $\mathrm{U} \otimes \mathrm{V}$. We have an action of $\mathrm{GL}(\mathrm{U}) \times \mathrm{GL}(\mathrm{V})$ on $\mathrm{U} \otimes \mathrm{V}$ which extends to an action on $\mathrm{S}(\mathrm{U} \otimes \mathrm{V})$.

Proposition 2.3. *As a* $\mathrm{GL}(\mathrm{U}) \times \mathrm{GL}(\mathrm{V})$*-module, the algebra* $\mathrm{S}(\mathrm{U} \otimes \mathrm{V})$ *can be decomposed as:*

$$\mathrm{S}(\mathrm{U} \otimes \mathrm{V}) = \bigoplus_{\mathrm{D} \in \Lambda_{\mathrm{U},\mathrm{V}}} \rho_{\mathrm{U}}^{\mathrm{D}} \otimes \rho_{\mathrm{V}}^{\mathrm{D}}\ ,$$

where $\Lambda_{\mathrm{U},\mathrm{V}}$ *is the set of all Young diagrams of depth at most* $\min(\dim_{\mathbb{C}}(\mathrm{U}), \dim_{\mathbb{C}}(\mathrm{V}))$.

Before proving the proposition, we recall a technical lemma.

Lemma 2.4. *(1) Let* G *be a group and* U_1 *and* U_2 *be two irreducible representations of* G*. Then, the space* $(\mathrm{U}_1 \otimes \mathrm{U}_2^*)^{\Delta(\mathrm{G}\times\mathrm{G})}$ *is non-trivial if and only if the* G*-modules* U_1 *and* U_2^* *are equivalent. More precisely, if the two* G*-modules are equivalent, the dimension of* $(\mathrm{U}_1 \otimes \mathrm{U}_2^*)^{\Delta(\mathrm{G}\times\mathrm{G})}$ *is* 1*.*

(2) The representations of the symmetric group are self-dual.

Proof. (1) Let $\Psi : \mathrm{U}_1 \otimes \mathrm{U}_2^* \to \mathrm{Hom}(\mathrm{U}_2, \mathrm{U}_1)$ be the map defined by:

$$\Psi(u_1 \otimes u_2^*)(u_2) = u_2^*(u_2) u_1\ , \qquad \left(u_1 \in \mathrm{U}_1, u_2 \in \mathrm{U}_2, u_2^* \in \mathrm{U}_2^*\right)\ .$$

This map is an isomorphism of G-modules. In particular, we have

$$\Psi\left((\mathrm{U}_1 \otimes \mathrm{U}_2^*)^{\Delta(\mathrm{G}\times\mathrm{G})}\right) = \mathrm{Hom}_{\mathrm{G}}(\mathrm{U}_2, \mathrm{U}_1)\ .$$

According to Schur's Lemma, $\mathrm{Hom}_{\mathrm{G}}(\mathrm{U}_2, \mathrm{U}_1)$ is non-zero (more precisely one-dimensional) if and only if U_1 and U_2 are equivalent.

(2) See [10]. □

Proof of Proposition 2.3. We have a natural $\mathbb{Z}$-gradation on the symmetric algebra

$$\mathrm{S}(\mathrm{U}\otimes\mathrm{V})=\bigoplus_{d=0}^{\infty}\mathrm{S}^{d}(\mathrm{U}\otimes\mathrm{V})$$

inherited from the one on $\mathrm{T}(\mathrm{U}\otimes\mathrm{V})$. Obviously, every component of this direct sum is a $\mathrm{GL}(\mathrm{U})\times\mathrm{GL}(\mathrm{V})$-module and

$$\mathrm{S}^{d}(\mathrm{U}\otimes\mathrm{V})=[(\mathrm{U}\otimes\mathrm{V})^{d}]^{\mathscr{S}_d}\,,$$

where $[(\mathrm{U}\otimes\mathrm{V})^{d}]^{\mathscr{S}_d}$ is the set of $\mathscr{S}_d$-invariants for the action of $\mathscr{S}_d$ on $(\mathrm{U}\otimes\mathrm{V})^{d}$. Then,

$$\begin{aligned}\mathrm{S}^{d}(\mathrm{U}\otimes\mathrm{V}) &= [(\mathrm{U}\otimes\mathrm{V})^{d}]^{\mathscr{S}_d}=\left(\mathrm{U}^{\otimes d}\otimes\mathrm{V}^{\otimes d}\right)^{\Delta(\mathscr{S}_d\times\mathscr{S}_d)}\\ &= \left[\left(\bigoplus_{\mathrm{D}_1}\rho_{\mathrm{U}}^{\mathrm{D}_1}\otimes\sigma^{\mathrm{D}_1}\right)\otimes\left(\bigoplus_{\mathrm{D}_2}\rho_{\mathrm{V}}^{\mathrm{D}_2}\otimes\sigma^{\mathrm{D}_2}\right)\right]^{\Delta(\mathscr{S}_d\times\mathscr{S}_d)}\\ &= \bigoplus_{\mathrm{D}_1,\mathrm{D}_2}\rho_{\mathrm{U}}^{\mathrm{D}_1}\otimes\rho_{\mathrm{V}}^{\mathrm{D}_2}\otimes\left(\sigma^{\mathrm{D}_1}\otimes\sigma^{\mathrm{D}_2}\right)^{\Delta(\mathscr{S}_d\times\mathscr{S}_d)}\,.\end{aligned}$$

It follows from Lemma 2.4 that

$$\mathrm{S}^{d}(\mathrm{U}\otimes\mathrm{V})=\bigoplus_{\mathrm{D}}\rho_{\mathrm{U}}^{\mathrm{D}}\otimes\rho_{\mathrm{V}}^{\mathrm{D}},$$

where D varies over all diagrams of size d and of depth at most $\min(\dim_{\mathbb{C}}(\mathrm{U}),\dim_{\mathbb{C}}(\mathrm{V}))$.

□

An other natural duality can be obtained by looking at the joint of $\mathrm{GL}(\mathrm{U})\times\mathrm{GL}(\mathrm{V})$ on the exterior algebra $\Lambda(\mathrm{U}\otimes\mathrm{V})$.

Notation 2.5. For a Young diagram D, we denote by D^{t} its transpose, i.e the rows (resp. columns) of D are the columns (resp. rows) of D^{t}. Obviously, if $\mathrm{D}\in\mathscr{C}^{+}$, D^{t} is not in general in $\mathrm{D}\in\mathscr{C}^{+}$.

Proposition 2.6. *As a* $\mathrm{GL}(\mathrm{U})\times\mathrm{GL}(\mathrm{V})$*-module, the algebra* $\mathrm{S}(\mathrm{U}\otimes\mathrm{V})$ *can be decomposed as:*

$$\Lambda(\mathrm{U}\otimes\mathrm{V})=\bigoplus_{\mathrm{D}\in\Gamma_{\mathrm{U},\mathrm{V}}}\rho_{\mathrm{U}}^{\mathrm{D}}\otimes\rho_{\mathrm{V}}^{\mathrm{D}^{t}}\,,$$

where $\Gamma_{\mathrm{U},\mathrm{V}}$ *is the set of all Young diagrams with at most* $\dim_{\mathbb{C}}(\mathrm{U})$ *rows and with rows lengths at most* $\dim_{\mathbb{C}}(\mathrm{V})$.

Remark 2.7. Let n be a positive integer. We denote by sgn_n the action of $\mathscr{S}_n$ on $\mathbb{C}^{n}$ given by:

(2) $$\mathrm{sgn}_n(\sigma)(v_1\otimes\ldots\otimes v_n)=\varepsilon(\sigma)v_{\sigma(1)}\otimes\ldots\otimes v_{\sigma(n)}\,,\qquad(\sigma\in\mathscr{S}_n,v_1,\ldots,v_n\in\mathrm{V})\;.$$

As proved in [2, Appendix A, Equation (A.29)], we have, for every Young tableau D, the following isomorphism of $\mathscr{S}_n$-modules:

(3) $$\sigma^{\mathrm{D}^{t}}\approx\sigma^{\mathrm{D}}\otimes\mathrm{sgn}_n\,,$$

where σ^{D} is the $\mathscr{S}_n$-module corresponding to D.

Proof. We will imitate the proof of Proposition 2.3. The group $\mathrm{GL}(\mathrm{U})\times\mathrm{GL}(\mathrm{V})$ acts on $\Lambda^{d}(\mathrm{U}\otimes\mathrm{V})$ and we have:

$$\Lambda^{d}(\mathrm{U}\otimes\mathrm{V})=[(\mathrm{U}\otimes\mathrm{V})^{\otimes d}]^{\mathrm{sgn}_d}\,.$$

Then, we get:

$$
\begin{aligned}
\Lambda^d(\mathrm{U}\otimes\mathrm{V}) &= [(\mathrm{U}\otimes\mathrm{V})^{\otimes d}]^{\mathrm{sgn}_d} = \left(\mathrm{U}^{\otimes d}\otimes\mathrm{V}^{\otimes d}\right)^{\Delta(\mathrm{sgn}_d\times\mathrm{sgn}_d)} \\
&= \left[\left(\bigoplus_{\mathrm{D}_1}\rho_{\mathrm{U}}^{\mathrm{D}_1}\otimes\sigma^{\mathrm{D}_1}\right)\otimes\left(\bigoplus_{\mathrm{D}_2}\rho_{\mathrm{V}}^{\mathrm{D}_2}\otimes\sigma^{\mathrm{D}_2}\right)\right]^{\Delta(\mathrm{sgn}_d\times\mathrm{sgn}_d)} \\
&= \bigoplus_{\mathrm{D}_1,\mathrm{D}_2}\rho_{\mathrm{U}}^{\mathrm{D}_1}\otimes\rho_{\mathrm{V}}^{\mathrm{D}_2}\otimes\left(\sigma^{\mathrm{D}_1}\otimes\sigma^{\mathrm{D}_2}\right)^{\Delta(\mathrm{sgn}_d\times\mathrm{sgn}_d)} \\
&= \bigoplus_{\mathrm{D}_1,\mathrm{D}_2}\rho_{\mathrm{U}}^{\mathrm{D}_1}\otimes\rho_{\mathrm{V}}^{\mathrm{D}_2}\otimes\left(\sigma^{\mathrm{D}_1}\otimes\sigma^{\mathrm{D}_2}\otimes\mathrm{sgn}_d\right)^{\Delta(\mathscr{S}_d\times\mathscr{S}_d)} \\
&= \bigoplus_{\mathrm{D}_1,\mathrm{D}_2}\rho_{\mathrm{U}}^{\mathrm{D}_1}\otimes\rho_{\mathrm{V}}^{\mathrm{D}_2}\otimes\left(\sigma^{\mathrm{D}_1}\otimes\sigma^{\mathrm{D}_2^t}\right)^{\Delta(\mathscr{S}_d\times\mathscr{S}_d)} ,
\end{aligned}
$$

and we conclude using Equation (3) and Lemma 2.4.

□

Using the previous results, we can deduce a result on invariants that we will use later in Section 3. Let U, V and W three finite dimensional vector spaces over $\mathbb{C}$. We have an action of GL(U) on the space $\mathrm{U}\otimes\mathrm{V}\oplus\mathrm{U}^*\otimes\mathrm{W}$ which can be extended to an action of GL(U) on $\Lambda(\mathrm{U}\otimes\mathrm{V}\oplus\mathrm{U}^*\otimes\mathrm{W})$. Now arise the question of an explicit determination of generators of the algebra $\Lambda(\mathrm{U}\otimes\mathrm{V}\oplus\mathrm{U}^*\otimes\mathrm{W})^{\mathrm{GL(U)}}$. Let $\{u_1,\dots,u_n\}$ be a basis of U, $\{u_1^*,\dots,u_n^*\}$ be the dual basis of U^*, $\{v_1,\dots,v_m\}$ be a basis of V and $\{w_1,\dots,w_k\}$ be a basis of W. Using that the GL(U)-invariants on $\mathrm{U}\otimes\mathrm{U}^*$ are colinear to

$$\sum_{i=1}^n u_i\otimes u_i^* ,$$

we got that the elements $\xi_{a,b}$, $1\le a\le m, 1\le b\le k$, given by

$$\xi_{a,b} = \sum_{i=1}^n u_i\otimes v_a\wedge u_i^*\otimes w_b , \tag{4}$$

are in $\Lambda(\mathrm{U}\otimes\mathrm{V}\oplus\mathrm{U}^*\otimes\mathrm{W})^{\mathrm{GL(U)}}$. In [9], R. Howe proved the following result.

Theorem 2.8. *The algebra* $\Lambda(\mathrm{U}\otimes\mathrm{V}\oplus\mathrm{U}^*\otimes\mathrm{W})^{\mathrm{GL(U)}}$ *is generated by the degree* 2 *elements* $\xi_{a,b}$ *given in Equation* (4).

Sketch of the proof. We have $(\mathrm{U}\otimes\mathrm{V}\otimes\mathrm{U}^*\otimes\mathrm{W})^{\mathrm{GL(U)}}\approx\mathrm{V}\otimes\mathrm{W}$. In particular, we get an homomorphism:

$$\mathrm{S}(\mathrm{U}\otimes\mathrm{W})\to\Lambda(\mathrm{V}\otimes\mathrm{U}\oplus\mathrm{V}^*\otimes\mathrm{W})^{\mathrm{GL(V)}} . \tag{5}$$

The space $\Lambda(\mathrm{U}\otimes\mathrm{V}\oplus\mathrm{U}^*\otimes\mathrm{W})^{\mathrm{GL(U)}}$ has a natural structure of $\mathrm{GL(V)}\times\mathrm{GL(W)}$-module. We can give a precise structure of this module. Indeed, we have:

$$
\begin{aligned}
\Lambda(\mathrm{U}\otimes\mathrm{V}\oplus\mathrm{U}^*\otimes\mathrm{W})^{\mathrm{GL(U)}} &= \left(\Lambda(\mathrm{U}\otimes\mathrm{V})\otimes\Lambda(\mathrm{U}^*\otimes\mathrm{W})\right)^{\mathrm{GL(U)}} \\
&= \left(\left(\sum_{\mathrm{D}}\rho_{\mathrm{U}}^{\mathrm{D}}\otimes\rho_{\mathrm{V}}^{\mathrm{D}}\right)\otimes\left(\sum_{\mathrm{E}}\rho_{\mathrm{U}^*}^{\mathrm{E}}\otimes\rho_{\mathrm{W}}^{\mathrm{E}}\right)\right)^{\mathrm{GL(U)}} \\
&= \sum_{\mathrm{D}}\sum_{\mathrm{E}}\left(\rho_{\mathrm{U}}^{\mathrm{D}}\otimes\rho_{\mathrm{U}^*}^{\mathrm{E}}\right)^{\mathrm{GL(U)}}\otimes\rho_{\mathrm{V}}^{\mathrm{D}}\otimes\rho_{\mathrm{W}}^{\mathrm{E}} = \sum_{\mathrm{D}}\rho_{\mathrm{V}}^{\mathrm{D}}\otimes\rho_{\mathrm{W}}^{\mathrm{D}^t} .
\end{aligned}
$$

Obviously, the map given in Equation (5) is an homomorphism of $\mathrm{GL(V)} \times \mathrm{GL(W)}$-modules. According to Proposition 2.3, we have to prove that the map given in Equation (5) is surjective.
For that, we have to prove that the highest weight of the modules appearing in the last equality can be obtained using the elements $\xi_{a,b}$. For this last part, check [9, Proof Theorem 4.2.3] (or [9, Proof Theorem 2.2.2]).

□

3. Pin-duality for the general linear group

Let E be a vector space over $\mathbb{R}$ or $\mathbb{C}$ endowed with a non-degenerate, symmetric, bilinear form b. We denote by T(E) the tensor algebra of E and by $\mathscr{T}$ the ideal generated by the elements of the form $x \otimes y + y \otimes x - \mathrm{b}(x, y).1, x, y \in \mathrm{E}$. The Clifford algebra, denoted by Cliff(E, b), is defined by

$$\mathrm{Cliff(E, b)} = \mathrm{T(E)}/\mathscr{T} .$$

The Clifford algebra inherits a natural structure of $\mathbb{Z}_2$-graded algebra coming from the gradation on T(E). The natural composition $\mathrm{E} \to \mathrm{T(E)} \to \mathrm{Cliff(E, b)}$ is an injection and we will identify the element $e \in \mathrm{E}$ with its image in Cliff(E, b). If we fix an orthonormal basis $\{e_1, \ldots, e_n\}$ of E, the elements $e_{i_1} e_{i_2} \ldots e_{i_k}$, $1 \leq i_1 < \ldots < i_k \leq n$ form a basis of the algebra Cliff(E, b). For more details, one can check [11, Chapter 2], [13, Chapter 5] or [3, Chapter 6.1].
On Cliff(E, b), we define the following involution α:

$$\alpha(e_1 \ldots e_k) = (-1)^k e_1 \ldots e_k ,$$

and the antiautomorphism τ

$$\tau(e_1 \ldots e_k) = e_k \ldots e_1 .$$

If E is real, the inclusion $\mathrm{E} \subseteq \mathrm{E}_{\mathbb{C}}$ naturally extends to an inclusion $\mathrm{Cliff(E, b)} \subseteq \mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})$, where $\mathrm{E}_{\mathbb{C}}$ and $\mathrm{b}_{\mathbb{C}}$ denote the complexifications of E and b respectively.
From now on, we assume that E is a vector space over $\mathbb{R}$. We denote by Pin(E, b) the subspace of Cliff(E, b) given by:

$$\mathrm{Pin(E, b)} = \left\{ x \in \mathrm{Cliff(E, b)} : x.\tau(x) = \pm 1 \text{ and } \alpha(x)\gamma(\mathrm{E})x^{-1} = \gamma(\mathrm{E}) \right\} . \tag{6}$$

One can check easily that the map $\pi : \mathrm{Pin(E, b)} \to \mathrm{O(E, b)}$ given by:

$$\pi(x)(e) = \alpha(x)ex^{-1} , \qquad (x \in \mathrm{Pin(E, b)}, e \in \mathrm{E}) ,$$

is well-defined and surjective with kernel $\mathrm{Ker}(\pi) = \{\pm 1\}$. Remark that when x is not isotropic $\pi(x)$ is the reflexion in the hyperplane $x^{\perp}$.
We denote by T the well-known quantization map ([13, Section 5.3], see also [11, Section 2.2.5]) :

$$\mathrm{T} : \Lambda(\mathrm{E}) \ni v_1 \wedge \cdots \wedge v_k \to \frac{1}{k!} \sum_{\sigma \in \mathscr{S}_k} \epsilon(\sigma) v_{\sigma(1)} \cdots v_{\sigma(k)} \in \mathrm{Cliff(E, b)} . \tag{7}$$

One can easily check that T is an isomorphism of Pin(E, b)-modules for the natural Pin(E, b)-action on both sides. One should note that the image of degree 2 elements $\Lambda^2(\mathrm{E})$ via this map is the Lie algebra $\mathfrak{pin}(\mathrm{E}, \mathrm{b})$ of Pin(E, b) (see [11, Section 2.2.10]).
We recall (see [11, Section 3]) the definition of the complex spinorial representation for a real even dimensional vector space E endowed with a non-degenerate symmetric bilinear form (the construction is slightly different when E is odd dimensional).

Definition 3.1. Let (E, b) be a real even dimensional vector space endowed with a non-degenerate symmetric bilinear form. Let F be a Lagrangian (maximal isotropic) subspace of $(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})$ (in particular, the space $\mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})/\mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})\mathrm{F}$ is a $\mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})$-module). The induced $\mathrm{Pin}(\mathrm{E}, \mathrm{b})$-representation is called the complex spinorial representation.

The complex spinorial representation is irreducible and up to equivariant isomorphism does not depend on the choice of the Lagrangian F. In the rest of the document, this representation will be denoted by (Π, S).

3.1. **A general method.** We start this section by stating a theorem that we will not proven here.

Theorem 3.2. *Let* $(\mathrm{G} = \mathrm{GL}(n, \mathbb{K}), \mathrm{G}' = \mathrm{GL}(m, \mathbb{K}))$ *be the dual pair in* $\mathrm{O}(\mathrm{E}, \mathrm{b})$*, where* $\mathbb{K} = \mathbb{R}, \mathbb{C}$ *or* $\mathbb{H}$*,* $\mathrm{E} = (\mathbb{K}^n \otimes \mathbb{K}^m)_{\mathbb{R}} \oplus (\mathbb{K}^n \otimes \mathbb{K}^m)^*_{\mathbb{R}}$ *and* b *is given by*

$$\mathrm{b}(e_1 + e_1^*, e_2 + e_2^*) = e_1^*(e_2) + e_2^*(e_1) .$$

Then, the preimages $(\widetilde{\mathrm{GL}}(n, \mathbb{K}), \widetilde{\mathrm{GL}}(m, \mathbb{K}))$ *form a dual pair in* $\mathrm{Pin}(\mathrm{E}, \mathrm{b})$*.*

Proof. See [4, Section 3].

□

Let (E, b) be an even dimensional vector space over $\mathbb{R}$ and let (Π, S) be the corresponding complex spinorial representation of $\mathrm{Pin}(\mathrm{E}, \mathrm{b})$ defined above. As before, we denote by $\mathrm{E}_{\mathbb{C}}$ (resp. $\mathrm{b}_{\mathbb{C}}$) the complexification of E (resp. b). The goal is to prove that for the dual pairs $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}')$ of $\mathrm{Pin}(\mathrm{E}, \mathrm{b})$ considered in Theorem 3.2, the representation (Π, S) sets up a Howe correspondance i.e.

$$\Pi = \bigoplus_{\lambda \in \hat{\mathrm{G}}_{\Pi}} \lambda \otimes \theta(\lambda) ,$$

where $\hat{\mathrm{G}}_{\Pi}$ is the set of equivalence classes of irreducible finite dimensional representations (Π, V_{Π}) of $\widetilde{\mathrm{G}}$ such that $\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathrm{V}_{\Pi}, \mathrm{S}) \neq \{0\}$ and θ is an injective map associating an irreducible representation of $\widetilde{\mathrm{G}}'$ to an irreducible representation of $\widetilde{\mathrm{G}}$.
According to the double commutant theorem (see Appendix A), this arises if and only if we have the following equality of algebras

$$\mathrm{Comm}_{\mathrm{End(S)}} \left\langle \Pi(\widetilde{\mathrm{G}}) \right\rangle = \left\langle \Pi(\widetilde{\mathrm{G}}') \right\rangle . \tag{8}$$

Let $\widetilde{\Pi} : \mathrm{Pin}(\mathrm{E}, \mathrm{b}) \to \mathrm{GL}(\mathrm{End}(\mathrm{S}))$ be the representation we have by conjugation:

$$\widetilde{\Pi}(c)(X) = \Pi(c)X\Pi(c)^{-1} , \qquad (c \in \mathrm{Pin}(\mathrm{E}, \mathrm{b}), \mathrm{X} \in \mathrm{End}(\mathrm{S})) ,$$

and $\widetilde{\rho} : \mathrm{Pin}(\mathrm{E}, \mathrm{b}) \to \mathrm{GL}(\mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}}))$ be the representation given by:

$$\widetilde{\rho}(c)(X) = cXc^{-1} , \qquad (c \in \mathrm{Pin}(\mathrm{E}, \mathrm{b}), X \in \mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})) .$$

Obviously, both $\mathrm{Pin}(\mathrm{E}, \mathrm{b})$-modules $\widetilde{\Pi}$ and $\widetilde{\rho}$ are equivalent. We denote by $\widetilde{\gamma}$ the isomorphism between $\mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})$ and $\mathrm{End}(\mathrm{S})$. Then, for every subgroup G of $\mathrm{O}(\mathrm{E}, \mathrm{b})$, we have:

$$\widetilde{\gamma}\left(\mathrm{End}(\mathrm{S})^{\widetilde{\mathrm{G}}}\right) = \mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})^{\widetilde{\mathrm{G}}} , \tag{9}$$

Using the map T given in Equation (7) and the fact that the actions of $\widetilde{\mathrm{G}}$ on $\mathrm{End}(\mathrm{S})$, $\mathrm{Cliff}(\mathrm{E}_{\mathbb{C}}, \mathrm{b}_{\mathbb{C}})$ and $\Lambda(\mathrm{E}_{\mathbb{C}})$ factor through G, we get:

$$\mathrm{End}(\mathrm{S})^{\widetilde{\mathrm{G}}} = \widetilde{\gamma}^{-1} \circ \mathrm{T}\left(\Lambda(\mathrm{E}_{\mathbb{C}})^{\widetilde{\mathrm{G}}}\right) = \widetilde{\gamma}^{-1} \circ \mathrm{T}(\Lambda(\mathrm{E}_{\mathbb{C}})^{\mathrm{G}}) . \tag{10}$$

Whence, up to the previous isomorphisms, we need to prove that $\Lambda(\mathrm{E}_{\mathbb{C}})^{\mathrm{G}}$ is spanned, as an algebra, by the image of G'.

3.2. **Real general linear group.** According to Lemma B.3, we got that:

$$\Lambda(\mathrm{E}_{\mathbb{C}})^{\mathrm{GL}(n,\mathbb{R})} = \Lambda(\mathrm{E}_{\mathbb{C}})^{\mathrm{GL}(n,\mathbb{C})} = \Lambda(\mathbb{C}^n \otimes \mathbb{C}^m \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)^*)^{\mathrm{GL}(n,\mathbb{C})}. \tag{11}$$

Then, according to theorem 2.8, we get:

$$\begin{aligned}\Lambda(\mathrm{E}_{\mathbb{C}})^{\mathrm{GL}(n,\mathbb{R})} &= \Lambda(\mathbb{C}^n \otimes \mathbb{C}^m \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)^*)^{\mathrm{GL}(n,\mathbb{C})} \\ &= \left\langle \Lambda^2(\mathbb{C}^n \otimes \mathbb{C}^m \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)^*)^{\mathrm{GL}(n,\mathbb{C})} \right\rangle \\ &= \left\langle (\mathbb{C}^n \otimes \mathbb{C}^m \otimes (\mathbb{C}^n \otimes \mathbb{C}^m)^*)^{\mathrm{GL}(n,\mathbb{C})} \right\rangle \\ &= \left\langle \mathfrak{gl}(\mathbb{C}^n \otimes \mathbb{C}^m)^{\mathrm{GL}(n,\mathbb{C})} \right\rangle \\ &= \langle \mathfrak{gl}(\mathbb{C}^m) \rangle .\end{aligned}$$

3.3. **Complex and quaternionic general linear group.** In this section, we want to imitate what we did for the real general linear group. In particular, according to Lemma B.3 and Equation (11), we have to determine the complexification of the real algebraic groups $\mathrm{GL}(n,\mathbb{C})$ and $\mathrm{GL}(n,\mathbb{H})$.
We first recall the definition of the complexification of a Lie group.

Definition 3.3. If G is a real algebraic group, the complexification of G is a pair $(\mathrm{G}_{\mathbb{C}},\gamma)$, where G is a complex Lie group and $\gamma : \mathrm{G} \to \mathrm{G}_{\mathbb{C}}$ is a $\mathbb{R}$-morphism analytic map such that for every complex Lie group H and every $\mathbb{R}$-morphism $\phi : \mathrm{G} \to \mathrm{H}$, there exists a unique $\mathbb{C}$-analytic map $\psi : \mathrm{G}_{\mathbb{C}} \to \mathrm{H}$ such that $\phi = \psi \circ \gamma$ (one can check [1, Chapter 3, 6.10]).

Let's start with $\mathrm{GL}(n,\mathbb{C})$. We denote by $\Psi_n : \mathrm{Mat}(n,\mathbb{C}) \to \mathrm{Mat}(2n,\mathbb{R})$ the map given by:

$$\Psi_n(\mathrm{A}+i\mathrm{B}) = \begin{pmatrix} \mathrm{A} & -\mathrm{B} \\ \mathrm{B} & \mathrm{A} \end{pmatrix} \qquad (\mathrm{A},\mathrm{B} \in \mathrm{Mat}(n,\mathbb{R})).$$

We denote by $\mathrm{G}_n = \Psi_n(\mathrm{GL}(n,\mathbb{C}))$ the image of $\mathrm{GL}(n,\mathbb{C})$. We have:

$$\mathrm{G}_n = \{\mathrm{X} \in \mathrm{GL}(2n,\mathbb{R}), \mathrm{X}\mathrm{J}_n = \mathrm{J}_n\mathrm{X}\} ,$$

where $\mathrm{J}_n = \begin{pmatrix} 0 & -\mathrm{Id}_n \\ \mathrm{Id}_n & 0 \end{pmatrix}$. Then,

$$\mathrm{G}_n^{\mathbb{C}} = \{\mathrm{X} \in \mathrm{GL}(2n,\mathbb{C}), \mathrm{X}\mathrm{J}_n = \mathrm{J}_n\mathrm{X}\} .$$

Then matrix J_n is diagonalisable over $\mathbb{C}$; more precisely,

$$\mathrm{J}_n = \mathrm{P}_n\mathrm{K}_n\mathrm{P}_n^{-1} \qquad \mathrm{K}_n = \begin{pmatrix} i\mathrm{Id}_n & 0 \\ 0 & -i\mathrm{Id}_n \end{pmatrix}.$$

Then, we get that:

$$\mathrm{G}_n^{\mathbb{C}} = \mathrm{P}_n^{-1}\{\mathrm{C} \in \mathrm{GL}(2n,\mathbb{C}), \mathrm{C}\mathrm{K}_n = \mathrm{K}_n\mathrm{C}\}\,\mathrm{P}_n .$$

Obviously,

$$\{\mathrm{C} \in \mathrm{GL}(2n,\mathbb{C}), \mathrm{C}\mathrm{K}_n = \mathrm{K}_n\mathrm{C}\} = \left\{ \mathrm{C} = \begin{pmatrix} \mathrm{C}_1 & 0 \\ 0 & \mathrm{C}_2 \end{pmatrix}, \mathrm{C}_1,\mathrm{C}_2 \in \mathrm{GL}(n,\mathbb{C}) \right\} ,$$

and by changing the basis, we see that the complexification $\mathrm{G}_n^{\mathbb{C}}$ of $\mathrm{GL}(n,\mathbb{C})$ is $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ and the map $\gamma : \mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(n,\mathbb{C}) \times \mathrm{GL}(n,\mathbb{C})$ is given by:

$$\gamma(g) = (g,\bar{g}) .$$

According to Appendix C, we have that $(\mathbb{C}^n \otimes \mathbb{C}^m)_{\mathbb{R}} \otimes \mathbb{C} = \mathbb{C}^n \otimes \mathbb{C}^m \oplus \mathbb{C}^n \otimes \mathbb{C}^m$. Moreover, the action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathbb{C}^n \otimes \mathbb{C}^m \oplus \mathbb{C}^n \otimes \mathbb{C}^m$ extend to an action of its complexification $\mathrm{GL}(n, \mathbb{C}) \times \mathrm{GL}(n, \mathbb{C})$ on the same space given by:

$$(g_1, g_2)(u_1 \otimes v_1 + u_2 \otimes v_2) = g_1(u_1) \otimes v_1 + g_2(u_2) \otimes v_2 \,.$$

We can now prove the duality for the dual pair $(\mathrm{G} = \mathrm{GL}(n, \mathbb{C}), \mathrm{G}' = \mathrm{GL}(m, \mathbb{C}))$. We have:

$$\begin{aligned}
\Lambda(\mathrm{E}_{\mathbb{C}})^{\mathrm{GL}(n,\mathbb{C})} &= \Lambda(\mathbb{C}^n \otimes \mathbb{C}^m \oplus \mathbb{C}^n \otimes \mathbb{C}^m \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)^* \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)^*)^{\mathrm{GL}(n,\mathbb{C})\times\mathrm{GL}(n,\mathbb{C})} \\
&= \Lambda(\mathbb{C}^n \otimes \mathbb{C}^m \oplus (\mathbb{C}^n \otimes \mathbb{C}^m)^*)^{\mathrm{GL}(n,\mathbb{C})} \otimes \Lambda(\mathbb{C}^n \otimes \mathbb{C}^m \oplus \mathbb{C}^n \otimes \mathbb{C}^m)^{\mathrm{GL}(n,\mathbb{C})} \\
&= \left\langle (\mathbb{C}^n \otimes \mathbb{C}^m \oplus \mathbb{C}^{n*} \otimes \mathbb{C}^{m*})^{\mathrm{GL}(n,\mathbb{C})} \right\rangle \otimes \left\langle (\mathbb{C}^n \otimes \mathbb{C}^m \oplus \mathbb{C}^{n*} \otimes \mathbb{C}^{m*})^{\mathrm{GL}(n,\mathbb{C})} \right\rangle \quad \text{(Theorem 2.8)} \\
&= \left\langle \mathfrak{gl}(\mathbb{C}^n \otimes \mathbb{C}^m)^{\mathrm{GL}(n,\mathbb{C})} \right\rangle \otimes \left\langle \mathfrak{gl}(\mathbb{C}^n \otimes \mathbb{C}^m)^{\mathrm{GL}(n,\mathbb{C})} \right\rangle \\
&= \langle \mathfrak{gl}(m, \mathbb{C})_{\mathbb{C}} \rangle = \langle \mathfrak{gl}(m, \mathbb{C}) \rangle \,.
\end{aligned}$$

For the dual pair $(\mathrm{G} = \mathrm{GL}(n, \mathbb{H}), \mathrm{G}' = \mathrm{GL}(m, \mathbb{H}))$, the computations are similar by using that

$$(\mathbb{H}^n \otimes \mathbb{H}^m)_{\mathbb{R}} \otimes \mathbb{C} \approx \mathbb{C}^{2n} \otimes \mathbb{C}^{2m} \,, \qquad \mathrm{GL}(n, \mathbb{H})_{\mathbb{C}} \approx \mathrm{GL}(2n, \mathbb{C}) \,.$$

Appendix A. Double commutant theorem

We work here in a simple situation: $\mathrm{G} \subseteq \mathrm{GL}(n, \mathbb{C})$ is a reductive linear algebraic group and (ρ, L) is a finite-dimensional G-module. We denote by $\mathscr{A}(\mathrm{G})$ the group algebra of G (here, ρ is not necessarily irreducible). Let $\hat{\mathrm{G}}$ be the set of irreducible representation of G. We denote by $\mathrm{Spec}(\rho)$ the set of $(\lambda, \mathrm{F}^\lambda) \in \hat{\mathrm{G}}$ such that $\mathrm{Hom}_{\mathrm{G}}(\mathrm{F}^\lambda, \mathrm{L}) \neq \{0\}$. Then, we have:

$$\mathrm{L} \approx \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} m_\lambda \mathrm{F}^\lambda \,, \tag{12}$$

where m_λ is the multiplicity of $(\lambda, \mathrm{F}^\lambda)$ in (L, ρ). More particularly, we have $m_\lambda = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{G}}(\mathrm{F}^\lambda, \mathrm{L})$ and Equation (12) can be written as

$$\mathrm{L} \approx \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} \mathrm{Hom}_{\mathrm{G}}(\mathrm{F}^\lambda, \mathrm{L}) \otimes \mathrm{F}^\lambda \,,$$

where G acts on $\mathrm{Hom}_{\mathrm{G}}(\mathrm{F}^\lambda, \mathrm{L}) \otimes \mathrm{F}^\lambda$ by $\mathrm{Id} \otimes \lambda(g)$.
Let $\mathscr{R} \subseteq \mathrm{End}(\mathrm{L})$ a subalgebra which satisfies the two following conditions:

(1) $\mathscr{R}$ acts irreducibly on L
(2) For all $g \in \mathrm{G}$ and $\mathrm{T} \in \mathscr{R}$, we have $\rho(g)\mathscr{R}\rho(g)^{-1} \in \mathscr{R}$.

Because $\dim(\mathrm{L}) < \infty$, we have $\mathscr{R} = \mathrm{End}(\mathrm{L})$ and because G acts on $\mathscr{R}$ by conjugation via ρ, it makes sense to consider the set of G-invariants in $\mathscr{R}$, i.e.

$$\mathscr{R}^{\mathrm{G}} = \left\{ \mathrm{T} \in \mathscr{R}, \rho(g)\mathrm{T}\rho(g)^{-1} = \mathrm{T} \, (\forall g \in \mathrm{G}) \right\} \,.$$

Viewing L as a module for $\mathscr{A}(\mathrm{G})$, we have that L is a $\mathscr{R}^{\mathrm{G}} \otimes \mathscr{A}(\mathrm{G})$-module (because $\mathscr{R}^{\mathrm{G}}$ and $\mathscr{A}(\mathrm{G})$ commute). More precisely, the space $\mathrm{Hom}_{\mathrm{G}}(\mathrm{F}^\lambda, \mathrm{L})$ has a natural structure of $\mathscr{R}^{\mathrm{G}}$-module and we get the following decomposition of L as a module of $\mathscr{R}^{\mathrm{G}} \otimes \mathscr{A}(\mathrm{G})$:

$$\mathrm{L} \approx \bigoplus_{\lambda \in \mathrm{Spec}(\rho)} \mathrm{Hom}_{\mathrm{G}}(\mathrm{F}^\lambda, \mathrm{L}) \otimes \mathrm{F}^\lambda \,.$$

We have the following proposition:

Proposition A.1. *We denote by* E^λ *the* $\mathscr{R}^{\mathrm{G}}$*-module* $\mathrm{Hom}_{\mathrm{G}}(\mathrm{F}^\lambda, \mathrm{L})$. *The map:*

$$\mathrm{F}^\lambda \to \mathrm{E}^\lambda$$

is a bijection between $\mathrm{Spec}(\rho)$ *and* $\mathrm{Spec}(\sigma)$*, where* $\mathrm{Spec}(\sigma)$ *is the set of equivalence classes of irreducible representation* $(\sigma_\lambda, \mathrm{E}^\lambda)$ *of* $\mathscr{R}^{\mathrm{G}}$ *s.t.* $\mathrm{Hom}_{\mathscr{R}^{\mathrm{G}}}(\mathrm{E}^\lambda, \mathrm{L}) \neq \{0\}$.

Appendix B. General results

Lemma B.1. *Let* (Π_1, V_1) *and* (Π_2, V_2) *two irreducible* G*-modules which are equivalent. We fix* T *in* $\mathrm{Hom}_G(\Pi_1, \Pi_2)$ *(unique up to a constant). Then,*

$$\mathrm{T}(\mathrm{Inv}(\mathrm{G}, \Pi_1, \mathrm{V}_1)) = \mathrm{Inv}(\mathrm{G}, \Pi_2, \mathrm{V}_2)\,.$$

Proof. Let v_1 be an element in $\mathrm{Inv}(\mathrm{G}, \Pi_1, \mathrm{V}_1)$. For all $g \in \mathrm{G}$, we have $\Pi_1(g)v_1 = v_1$. Because Π_1 and Π_2 are equivalent, we have $\Pi_1(g) = \mathrm{T}^{-1} \circ \Pi_2(g) \circ \mathrm{T}$. So,

$$\Pi_1(g)v_1 = v_1 \Leftrightarrow \mathrm{T}^{-1} \circ \Pi_2(g) \circ \mathrm{T}(v_1) = v_1 \Leftrightarrow \Pi_2(g)(\mathrm{T}(v_1)) = \mathrm{T}(v_1)\,.$$

In particular, $\mathrm{T}(v_1) \in \mathrm{Inv}(\mathrm{G}, \Pi_2, \mathrm{V}_2)$.
We prove easily the other inclusion.

□

Fix $(\mathrm{V}, \mathrm{b_V})$, where V is a real vector space and $\mathrm{b_V}$ a non-degenerate, symmetric, bilinear form on V, and let $(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C})$ be the corresponding complexification.

Lemma B.2. *Let* (Π, V) *be a finite dimensional representation of* $\mathrm{O}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C})$. *Then,*

$$\mathrm{Inv}(\mathrm{O}(\mathrm{V}_\mathbb{C}, b_\mathrm{V}^\mathbb{C}), \Pi, \mathrm{V}) = \mathrm{Inv}(\mathrm{O}(\mathrm{V}, \mathrm{b_V}), \Pi, \mathrm{V})\,.$$

Proof. Obviously, we have $\mathrm{Inv}(\mathrm{O}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C}), \Pi, \mathrm{V}) \subseteq \mathrm{Inv}(\mathrm{O}(\mathrm{V}, b_\mathrm{V}), \Pi, \mathrm{V})$. Let $v \in \mathrm{Inv}(\mathrm{O}(\mathrm{V}, \mathrm{b_V}), \Pi, \mathrm{V})$. Then, for all $g \in \mathrm{O}(\mathrm{V}, \mathrm{b_V})$, we have $\Pi(g)v = v$. In particular, for all $X \in \mathfrak{so}(\mathrm{V}, \mathrm{b_V})$, we have $d\Pi(X)v = 0$. Using that $\mathfrak{so}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C}) = \mathfrak{so}(\mathrm{V}, \mathrm{b_V}) \oplus i\mathfrak{so}(\mathrm{V}, \mathrm{b_V})$, we get that $d\Pi(X)v = 0$ for all $X \in \mathfrak{so}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C})$.
In particular, we get that $\Pi(g)v = v$ for all $g \in \mathrm{SO}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C})$. But $\mathrm{O}(\mathrm{V}, \mathrm{b_V}) \cap (\mathrm{O}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C}) \setminus \mathrm{SO}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C})) \neq \{\emptyset\}$, we get that $v \in \mathrm{Inv}(\mathrm{O}(\mathrm{V}_\mathbb{C}, \mathrm{b}_\mathrm{V}^\mathbb{C}), \Pi, \mathrm{V})$.

□

Similarly, we get the following lemma:

Lemma B.3. *Let* G *be an algebraic group over* $\mathbb{R}$*, and let* (Π, V) *be a regular representation of* $\mathrm{G}(\mathbb{C})$. *We denote by* $\mathrm{G}(\mathbb{C})_0$ *the connected component at identity of* $\mathrm{G}(\mathbb{C})$. *Assume that the intersection of each connected component of* $\mathrm{G}(\mathbb{R})$ *with* $\mathrm{G}(\mathbb{C})_0$ *is non empty. Then,*

$$\mathrm{Inv}(\mathrm{G}(\mathbb{R}), \Pi, \mathrm{V}) = \mathrm{Inv}(\mathrm{G}(\mathbb{C}), \Pi, \mathrm{V})\,.$$

Appendix C. Complexification of a complex vector space

Let U be a finite dimensional vector space over $\mathbb{R}$ and let $V = U \otimes_{\mathbb{R}} \mathbb{C}$. This is a vector space over $\mathbb{C}$ and

$$V = U \oplus iU\,.$$

Hence there is a conjugation on V,

$$c : U \oplus iU \ni u_1 + iu_2 \to u_1 - iu_2 \in U \oplus iU\,,$$

The map c is a bijective $\mathbb{C}$ anti-linear involution. The set of the fixed points coincides with $\mathrm{Fix}(V, c) = U$. Conversely, suppose V is a complex vector space and $c : V \to V$ is a bijective $\mathbb{C}$ anti-linear involution. Let $U = \mathrm{Fix}(V, c)$ be the set of the fixed points. Then V is a complexification of U, or equivalently, U is a real form of V, [3, 6.4.1]. Let V be a complex vector space. Then $V \oplus V$ is also a complex vector space and we'll see that we may view it as a complexification $(V_{\mathbb{R}})_{\mathbb{C}}$ of $V_{\mathbb{R}}$, the vector space V viewed as a vector space over $\mathbb{R}$. Indeed, let

$$c : V \oplus V \ni (u, v) \to (c(v), c(u)) \in V \oplus V\,.$$

Then c is a bijective $\mathbb{C}$ anti-linear involution and the set of the fixed points

$$\mathrm{Fix}(V \otimes V, c) = \{(u, c(u)),\ u \in V\}\ .$$

This set may be identified with V via the following $\mathbb{R}$ linear bijection

$$V \ni u \to (u, c(u)) \in \mathrm{Fix}(V \oplus V, c)\,.$$

Hence, $V \oplus V$ is a complexification of $V_{\mathbb{R}}$. We keep the notation introduced before. In particular V is a finite dimensional vector space over $\mathbb{C}$. The group GL(V) acts on V and we want to transfer it to an action on $V \oplus V$. Therefore we define a $\mathbb{C}$ anti-linear involution $c : \mathrm{GL(V)} \to \mathrm{GL(V)}$ by

$$c(g)v = c(g(c(v))) \qquad (g \in \mathrm{GL(V)},\ v \in V)\ .$$

Then

$$(gu, c(gu)) = (ug, c(g)c(u)) \qquad (g \in \mathrm{GL(V)}, u \in V)\ .$$

Hence the action we are looking for is

$$g(u, v) = (gu, c(g)v) \qquad (g \in \mathrm{GL(V)}, (u, v) \in V \oplus V)\ .$$

From here we see that the complexification $\mathrm{GL(V)}_{\mathbb{C}}$ of GL(V) viewed as a real algebraic group may be identified with $\mathrm{GL(V)} \times \mathrm{GL(V)}$ by

$$(g_1, g_2)(u, v) = (g_1u, g_2v) \qquad ((g_1, g_2) \in \mathrm{GL(V)} \times \mathrm{GL(V)},\ (u, v) \in V \oplus V)\ .$$

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