RESEARCH STATEMENT

ALLAN MERINO
(LAST UPDATED: MAY 1, 2022)

My research is in the field of representation theory of real reductive Lie groups. One of the main problems in representation theory is to determine the set of equivalence classes of irreducible unitary representations of a Lie group. Every irreducible unitary representation of a real reductive group is admissible, in the sense of Harish-Chandra. The irreducible admissible representations are understood in terms of Langlands classification. The building blocks in that theory are representations which occur in Harish-Chandra's Plancherel formula. They have large Gelfand-Kirillov dimension (equal to half the dimension of the wave front set of the representation).
On the other extreme are representations with small Gelfand-Kirillov dimension. Amongst them is the celebrated Weil representation of the metaplectic group. In contrast to Harish-Chandra and Langlands, Howe's theory of the local theta correspondence, or his theory of rank, suggests a way to organize the representations of classical groups, so that the building blocks have small Gelfand-Kirillov dimension.
Using the Weil representation, Howe established a one-to-one correspondence (known as the local theta correspondence) between particular representations of two subgroups $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$ forming a dual pair in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ (see [25]). This correspondence provides a nice way to construct unitary representations of small Gelfand-Kirillov dimension (see [31]). Moreover, a classification of all the irreducible highest weight representations of a classical group had been obtained in [11] by using Howe's correspondence.
I am interested in representation theory of Lie groups, Howe's correspondence and its applications to study some invariants attached to representations. More recently, I started working on a generalization of Howe's duality to Lie superalgebras and supergroups.

1. COMPLETED RESEARCH

My research focuses on two principal topics: the first one concerns the transfer of characters in the local theta correspondence ([37],[38], [39] and [40]) and the second one on the extension of Howe duality to the spinor-oscillator representation of the orthosymplectic Lie supergroup ([15] and [41]). In the following paragraph, I briefly recall the motivations and main background results.

1.1. Transfer of characters in the theta correspondence.

Let G be a real reductive Lie group and let $(\Pi, \mathscr{H})$ be a quasi-simple representation of G (see [17, Section 10]). For every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, the operator $\Pi(\Psi)$ defined by

$$\Pi(\Psi) = \int_{\mathrm{G}} \Psi(g)\Pi(g)dg\,,$$

where dg is a Haar measure on G, is well-defined and bounded on $\mathscr{H}$. In [16, Section 5], Harish-Chandra proved that for every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, $\Pi(\Psi)$ is a trace class operator and the corresponding map

$$\Theta_\Pi : \mathscr{C}_c^\infty(\mathrm{G}) \ni \Psi \to \mathrm{tr}(\Pi(\Psi)) \in \mathbb{C}$$

is a distribution in the sense of Laurent Schwartz (see [16, Section 5]). Moreover, in [19, Theorem 2], Harish-Chandra proved that there exists a locally integrable function Θ_Π on G, analytic on $\mathrm{G}^{\mathrm{reg}}$, such that $\Theta_\Pi = T_{\Theta_\Pi}$, i.e. for every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$,

$$\Theta_\Pi(\Psi) = \int_{\mathrm{G}} \Theta_\Pi(g)\Psi(g)dg.$$

The locally integrable function Θ_Π is called the character of the representation Π. In few cases, explicit formulas for Θ_Π had been found:

- If G is compact (Weyl).
- If $(\Pi, \mathscr{H})$ is a discrete series representation of G. More precisely, in [18] and [20], Harish-Chandra gave a classification of discrete series representations of G and determined the value of the corresponding character Θ_Π on the compact Cartan subgroup H of G. For holomorphic discrete series, the value of Θ_Π on the other Cartan subgroups had been determined by Hecht (see [21]). For a more geometric interpretation of the character for discrete series representations, see Rossmann's paper [50] where he proved a general conjecture of Kirillov [28] linking the character of Π and the Fourier transform of a co-adjoint orbit on G.
- If $(\Pi, \mathscr{H})$ an irreducible principal series representation (see [29, Proposition 10.18]).
- If $(\Pi, \mathscr{H})$ is an irreducible unitary highest weight representation, Enright's result (see [13, Corollary 2.3]) describes the restriction of the character Θ_Π to a maximal compact Cartan subgroup.

Let W be a finite dimensional vector space over $\mathbb{R}$ endowed with a non-degenerate, symmetric bilinear form $\langle\cdot,\cdot\rangle$. Let Sp(W) be the corresponding group of isometries of $(\mathrm{W}, \langle\cdot,\cdot\rangle)$, $\widetilde{\mathrm{Sp}}(\mathrm{W})$ be the metaplectic cover of Sp(W) (see [2, Definition 4.18]) and $(\omega, \mathscr{H})$ be the corresponding Weil representation (see [2, Section 4.8]). For an irreducible reductive dual pair $(\mathrm{G}, \mathrm{G}')$ in Sp(W), let $\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}'$ be the preimages of G and G′ in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ respectively. So, $(\widetilde{\mathrm{G}}, \widetilde{\mathrm{G}}')$ is a dual pair in $\widetilde{\mathrm{Sp}}(\mathrm{W})$ (see [25]).

For a subgroup $\widetilde{\mathrm{E}}$ of $\widetilde{\mathrm{Sp}}(\mathrm{W})$, we denote by $\mathscr{R}(\widetilde{\mathrm{E}}, \omega)$ the set of conjugacy classes of irreducible admissible representations $(\Pi, \mathscr{H}_\Pi)$ of $\widetilde{\mathrm{E}}$ which can be realized as a quotient of $\mathscr{H}^\infty$ by a closed $\omega^\infty(\widetilde{\mathrm{E}})$-invariant subspace. As proved by Howe (see [25, Theorem 1]), for every reductive dual pair $(\mathrm{G}, \mathrm{G}')$ of Sp(W), we have a one-to-one correspondence between $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ and $\mathscr{R}(\widetilde{\mathrm{G}}', \omega)$ whose graph is $\mathscr{R}(\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}}', \omega)$.

More precisely, if $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$, we denote by $\mathrm{N}(\Pi)$ the intersection of all the closed $\widetilde{\mathrm{G}}$-invariant subspaces $\mathscr{N}$ such that $\Pi \approx \mathscr{H}^\infty/\mathscr{N}$. Then the space $\mathscr{H}(\Pi) = \mathscr{H}^\infty/\mathrm{N}(\Pi)$ is a $\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}}'$-module and in particular $\mathscr{H}(\Pi) = \Pi \otimes \Pi'_1$, where Π'_1 is a $\widetilde{\mathrm{G}}'$-module, not irreducible in general, but Howe's duality theorem says that there exists a unique irreducible quotient Π' of Π'_1 with $\Pi' \in \mathscr{R}(\widetilde{\mathrm{G}}', \omega)$ and $\Pi \otimes \Pi' \in \mathscr{R}(\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}}', \omega)$.

Every representation appearing in the correspondence has a distribution character, and characters are analytic objects completely identifying the irreducible representations. I am interested in the transfer of characters in the Howe correspondence: how to link Θ_Π and $\Theta_{\Pi'}$ or more generally, how can we get $\Theta_{\Pi'}$ starting from Θ_Π?

Remark 1.1. When G is compact, the situation turns out to be much easier. Indeed, every $(\Pi, \mathscr{H}_\Pi) \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ is a subrepresentation of $(\omega, \mathscr{H})$ and in particular, its isotypic component

$\mathscr{H}(\Pi)$ is a closed subspace of $\mathscr{H}$. Moreover, $\widetilde{\mathrm{G}}'$ acts on $\mathscr{H}(\Pi)$ and we have $\mathscr{H}(\Pi) = \Pi \otimes \Pi'$, where Π' is an irreducible unitary representation of $\widetilde{\mathrm{G}}'$. We have:

$$\omega = \bigoplus_{\Pi \in \hat{\mathrm{G}}_\omega} \Pi \otimes \Pi' \,,$$

where $\hat{\mathrm{G}}_\omega$ is the set of irreducible representations $(\Pi, \mathscr{H}_\Pi)$ of $\widetilde{\mathrm{G}}$ such that $\mathrm{Hom}_{\widetilde{\mathrm{G}}}(\mathscr{H}_\Pi, \mathscr{H}) \neq \{0\}$ and where the sum is not an algebraic sum but the closure of the algebraic sum with respect to the topology of $\mathscr{H}$. This leads to character correspondence, as we shall see below.

Transfer of characters in the theta correspondence with one compact member (Journal of Lie Theory, 2020)

Let's fix a representation $(\Pi, \mathscr{H}_\Pi) \in \hat{\mathrm{G}}_\omega$ and denote by $\mathscr{P}_\Pi : \mathscr{H} \to \mathscr{H}(\Pi)$ the projection onto the Π-isotypic component as in Remark 1.1. For every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}}')$, we get:

$$\Theta_{\Pi'}(\Psi) = \mathrm{tr}\,(\mathscr{P}_\Pi \omega(\Psi)) = \mathrm{tr} \int_{\widetilde{\mathrm{G}}'} \left(\int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\tilde{g})} \omega(\tilde{g}\tilde{g}') d\tilde{g} \right) d\tilde{g}' = \int_{\widetilde{\mathrm{G}}'} \left(\int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\tilde{g})} \Theta(\tilde{g}\tilde{g}') d\tilde{g} \right) d\tilde{g}' \,,$$

where Θ is the character of ω (see [2, Definition 4.23]). Then by using the oscillator semigroup $\mathrm{Sp}(\mathrm{W}_\mathbb{C})^{++}$ introduced by Howe in [23] (see also [38, Section 3]), we get that the character $\Theta_{\Pi'}$ on $\widetilde{\mathrm{G}}'^{\mathrm{reg}}$ is given by:

$$\Theta_{\Pi'}(\widetilde{g'}) = \lim_{\substack{\widetilde{p} \to 1 \\ \tilde{p} \in \widetilde{\mathrm{G}}'^{++}}} \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})} \Theta(\widetilde{g g' p}) d\widetilde{g} \,, \qquad (\tilde{g}' \in \widetilde{\mathrm{G}}'^{\mathrm{reg}}) \,,$$

where $\mathrm{G}'^{++} = \mathrm{G}'_\mathbb{C} \cap \mathrm{Sp}(W_\mathbb{C})^{++}$ (see [38, Theorem 4.3]). In [38, Proposition 6.2], for $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(n), \mathrm{U}(p,q))$, we get an integral formula for the value of $\Theta_{\Pi'}$ on $\widetilde{\mathrm{H}}'^{\mathrm{reg}}$, where H' is the compact Cartan of G' and made explicit computations for $n = 1$ in [38, Proposition 6.4]. For the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(1), \mathrm{U}(1,1))$, we obtained in [38, Section 7] the value of the character $\Theta_{\Pi'}$ on the non-compact Cartan subgroup of G'. With a different method, we obtained similar results in [39, Appendix A] for $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(1), \mathrm{U}(p,q))$ on every Cartan subgroups of G' by using results of [4].

Characters of some unitary highest weight representations via the theta correspondence (Journal of Functional Analysis, 2020)

By a result of Przebinda ([46, Theorem 6.7]), we know that the pullback of the character $\Theta_{\Pi'}$ via the Cayley transform is given by

$$\tilde{c}_-^* \Theta_{\Pi'}(\varphi) = \mathrm{T}(\overline{\Theta_\Pi})(\phi) \,, \qquad (\varphi \in \mathscr{C}_c^\infty(\mathfrak{g}')) \,,$$

where $\tilde{c}_-$ is defined in [46, Equation 3.16] (or [37, Section 3]), $\mathrm{T} : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W})$ is the embedding of $\widetilde{\mathrm{Sp}}(\mathrm{W})$ in $\mathrm{S}^*(\mathrm{W})$ given in [2, Definition 4.23] (here, $\mathrm{S}^*(\mathrm{W})$ denotes the space of tempered distributions on W) and

$$\phi(w) = \mathscr{F}(\varphi\Theta \circ \tilde{c}) \circ \tau_{\mathfrak{g}'}(w) \,, \tag{1}$$

where $\mathscr{F} : \mathrm{S}(\mathfrak{g}') \mapsto \mathrm{S}(\mathfrak{g}'^*)$ is the Fourier transform defined in [46, Section 4], $\mathrm{S}(\mathfrak{g}')$ is the Schwartz algebra of $\mathfrak{g}'$ and $\mathfrak{g}'^*$ denotes the dual of $\mathfrak{g}'$.

One of the main technique in this paper to get the value of the character $\Theta_{\Pi'}$ was to use a result of Rossmann-Duflo-Vergne on the Fourier transform $\hat{\mu}_{\theta_\lambda}$ of a co-adjoint orbit $\theta_\lambda = \mathrm{Ad}^*(\mathrm{G})(\lambda)$ for a general parameter $\lambda \in \mathfrak{h}^*$, where $\mathfrak{h} = \mathrm{Lie}(\mathrm{H})$ and H a compact Cartan subgroup of G (the

fomula for $\hat{\mu}_{\theta_\lambda}$ had been obtained by Rossmann for a regular parameter λ in [28] and generalized by Duflo and Vergne in [10]).
I derived explicit, Weyl denominator free, formulas in [37, Theorem 5.11] for the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(n), \mathrm{U}(p,q))$, in [37, Theorem 6.7] for $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}(2n, \mathbb{R}), \mathrm{Sp}(2m, \mathbb{R}))$, in [37, Theorem 7.6] for $(\mathrm{G}, \mathrm{G}') = (\mathrm{O}(2n+1, \mathbb{R}), \mathrm{Sp}(2m, \mathbb{R}))$ and in [37, Theorem 8.12] for $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(n, \mathbb{H}), \mathrm{O}^*(m, \mathbb{H}))$. This is complementary to Enright's work [13].

The next two of my papers use the concept of Cauchy-Harish-Chandra integral introduced by T. Przebinda (see [48, Section 2]) in order to understand the transfer of characters in the correspondence for a general dual pair $(\mathrm{G}, \mathrm{G}')$. I recall in a few words the construction of this integral and the corresponding conjecture.

Let $\mathrm{T} : \widetilde{\mathrm{Sp}}(\mathrm{W}) \to \mathrm{S}^*(\mathrm{W})$ be the injection of the metaplectic group into the space of tempered distributions on W as in [2, Definition 4.23]. The map T can be extended to $\mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$ by

$$\mathrm{T}(\Psi) = \int_{\widetilde{\mathrm{Sp}}(\mathrm{W})} \Psi(\tilde{g})\mathrm{T}(\tilde{g})d\tilde{g}\,, \qquad (\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W})))\,,$$

where $d\tilde{g}$ is a Haar measure on $\widetilde{\mathrm{Sp}}(\mathrm{W})$. As proved in [2, Section 4.8], for every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{Sp}}(\mathrm{W}))$, $\mathrm{T}(\Psi) \in \mathrm{S}(\mathrm{W})$.
Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair $(\mathrm{G}, \mathrm{G}')$ in $\mathrm{Sp}(\mathrm{W})$ and $\mathrm{H}_1, \dots, \mathrm{H}_n$ be a maximal set of mutually non-conjugate Cartan subgroups of G. As explained in [55, Section 2.3.6], every Cartan subgroup H_i can be decomposed as $\mathrm{H}_i = \mathrm{T}_i\mathrm{A}_i$, where T_i is maximal compact in H_i. For every $1 \leq i \leq n$, we consider the subgroups $\mathrm{A}_i', \mathrm{A}_i''$ of $\mathrm{Sp}(\mathrm{W})$ given by $\mathrm{A}_i' = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}\mathrm{A}_i$ and $\mathrm{A}_i'' = \mathrm{C}_{\mathrm{Sp}(\mathrm{W})}\mathrm{A}_i'$. The pair $(\mathrm{A}_i', \mathrm{A}_i'')$ is a dual pair (not irreducible in general) in $\mathrm{Sp}(\mathrm{W})$.
For every $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{A}}_i')$, we define $\mathrm{Chc}(\Psi)$ by

$$\mathrm{Chc}(\Psi) = \int_{\mathrm{A}_i''\backslash \mathrm{W}_{\mathrm{A}_i''}} \mathrm{T}(\Psi)(w)\overline{dw}\,,$$

where $\mathrm{W}_{\mathrm{A}_i''}$ and $\overline{dw}$ are defined in [48, Section 1]. As proved in [48, Section 2], for every regular element $\tilde{h}_i \in \widetilde{\mathrm{H}}_i^{\mathrm{reg}}$, the pull-back of Chc through the map $\tau_{\tilde{h}_i} : \widetilde{\mathrm{G}}' \ni \tilde{g}' \to \tilde{h}_i\tilde{g}' \in \widetilde{\mathrm{A}}_i'$, denoted by $\mathrm{Chc}_{\tilde{h}_i}$, is well-defined. From now on, we assume that $\mathrm{rk}(\mathrm{G}) \leq \mathrm{rk}(\mathrm{G}')$. Using a result of Bouaziz on orbital integrals on reductive Lie groups (see [5]), Bernon and Przebinda constructed in [4] a map

$$\mathrm{Chc}^* : \mathscr{D}'(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}} \to \mathscr{D}'(\widetilde{\mathrm{G}}')^{\widetilde{\mathrm{G}}'}$$

such that for every $\widetilde{\mathrm{G}}$-invariant distribution Θ on $\widetilde{\mathrm{G}}$ given by a locally integrable function Θ on $\widetilde{\mathrm{G}}$,
(2)

$$\mathrm{Chc}^*(\Theta)(\Psi) = \sum_{i=1}^n \frac{1}{|\mathscr{W}(\mathrm{H}_i)|} \int_{\widetilde{\mathrm{H}}_i} \Theta(\tilde{h}_i)|\det(\mathrm{Id} - \mathrm{Ad}(\tilde{h}_i)^{-1})_{\mathfrak{g}/\mathfrak{h}_i}|^{\frac{1}{2}}\mathrm{Chc}_{\tilde{h}_i}(\Psi)d\tilde{h}_i\,, \qquad (\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}}'))\,.$$

Moreover, $\mathrm{Chc}^*(\mathrm{Eigen}(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}}) \subseteq \mathrm{Eigen}(\widetilde{\mathrm{G}}')^{\widetilde{\mathrm{G}}'}$, where $\mathrm{Eigen}(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}}$ is the set of $\widetilde{\mathrm{G}}$-invariant distributions on $\widetilde{\mathrm{G}}$ (see [19]). In [48], T. Przebinda conjectured the following:

Conjecture 1.2. Let G_1 and G_1' be the Zariski identity components of G and G′ respectively. Assume that $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ satisfies $\Theta_{\Pi|_{\widetilde{\mathrm{G}}\backslash\widetilde{\mathrm{G}_1}}} = 0$ if $\mathrm{G} = \mathrm{O}(p, q, \mathbb{K})$, $p, q \in \mathbb{Z}_+$ such that $p + q \in 2\mathbb{Z}$ and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Then, up to a constant, $\mathrm{Chc}^*(\Theta_\Pi) = \Theta_{\Pi_1'}$ on $\widetilde{\mathrm{G}}_1'$.

The conjecture is true if G is compact and was proven by Przebinda for unitary representations in the stable range case (see [49]).

Transfer of characters for discrete series representations of the unitary groups in the equal rank case via the Cauchy–Harish-Chandra integral, to appear in International Mathematics Research Notices

In this paper, I proved the Conjecture 1.2 for the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p,q), \mathrm{U}(r,s))$, with $p+q = r+s$ and Π a discrete series representation of $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$.
For this particular case, the Conjecture 1.2 can be stated slightly differently. First of all, according to Paul's result [44, Theorem 0.1], if we fix (p,q) and let r and s vary under the assumption that $p+q = r+s$, then for every genuine representation Π of $\widetilde{\mathrm{U}}(p,q)$, there exists a unique pair (r,s) such that $p+q = r+s$ and $\theta_{r,s}(\Pi) \neq 0$. Secondly, using a result of Li (see [32, Proposition 2.4]), if Π is a discrete series representation of $\widetilde{\mathrm{U}}(p,q)$, it can be embedded as a subrepresentation of ω and $\Pi' = \Pi'_1$. Moreover, by a result of Paul [45, Theorem 2.7], the corresponding representation $\Pi' = \theta_{r,s}(\Pi)$ is a discrete series representation of $\widetilde{\mathrm{G}'}$ and the correspondence of Harish-Chandra parameters for Π and Π' is well-known and explicit. In particular, $\Pi' = \Pi'_1$, $\mathrm{G} = \mathrm{G}_1$ and $\mathrm{G}' = \mathrm{G}'_1$. So the equality in the conjecture for this case can be rewritten, up to a constant, as $\mathrm{Chc}^*(\Theta_\Pi) = \Theta_{\Pi'}$.
In [18, Lemma 44], Harish-Chandra gave a parametrization of the discrete series characters under three conditions: expected shape on the compact Cartan subgroup, stays bounded when multiplied by Weyl denominator and has correct infinitesimal character. Using [4, Theorem 2.2] and [3, Theorem 0.9], I proved in [40] that $\mathrm{Chc}^*(\Theta_\Pi)$ verifies the three conditions of [18, Lemma 44] (see [40, Proposition 6.5], [40, Proposition 6.7] and [40, Lemma 6.10]) and obtain that $\mathrm{Chc}^*(\Theta_\Pi)$ is the character of a discrete series representations of $\widetilde{\mathrm{G}'}$. Then we conclude by using Paul's results on the correspondence of Harish-Chandra parameters (see [40, Corollary 6.11]) that $\mathrm{Chc}^*(\Theta_\Pi)$ is $\Theta_{\Pi'}$.
In [40], I also proved that the distribution $\mathrm{T}(\Theta_{\Pi^c})$ on the symplectic space W, where Π^c is the contragredient representation of Π, is well-defined (see [40, Proposition 7.3]) and such that $\mathrm{T}(\Theta_{\Pi^c}) = \mathrm{T}(\mathrm{Chc}^*(\Theta_{\Pi^c}))$ (see [40, Corollary 7.4]). In particular, we hope that the following diagram commutes (up to a constant) whenever the slanted arrows are well defined

$$\begin{array}{ccc} \mathscr{D}'(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}} & \xrightarrow{\mathrm{Chc}^*} & \mathscr{D}'(\widetilde{\mathrm{G}'})^{\widetilde{\mathrm{G}'}} \\ & {}_{\mathrm{T}}\searrow \quad \swarrow_{\mathrm{T}} & \\ & \mathrm{S}^*(\mathrm{W})^{\widetilde{\mathrm{G}}\cdot\widetilde{\mathrm{G}'}} & \end{array} \quad . \tag{3}$$

The operator $\omega(d_\Pi \Theta_{\Pi^c})$ is a well-defined projection operator onto the Π-isotypic component $\mathscr{H}$, where d_Π is the formal degree of Π (see [29, Chapter 9.3]). In particular, we recover a result of Xue (see [40, Section 7] and [56, Proposition 3.4])

$$\omega(d_\Pi \Theta_{\Pi^c}) = \omega(d_{\Pi'} \Theta_{\Pi'^c})\,, \tag{4}$$

which means that the slanted arrows in (3) agree in this case.

Characters of representations of $\mathrm{U}(n, n+1)$ via double lifting from $\mathrm{U}(1)$, to appear in Representation Theory

We first consider the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(1), \mathrm{U}(1,1))$. As explained in [31], for every representation $\Pi \in \widetilde{\mathrm{U}}(1)$, the corresponding representation Π'_1 is non-trivial and $\Pi'_1 = \Pi'$. The

value of the character $\Theta_{\Pi'}$ of Π' on both Cartan subgroups of $\widetilde{\mathrm{U}}(1,1)$ has been computed in [37]. We use another method for $(\mathrm{G},\mathrm{G}') = (\mathrm{U}(1),\mathrm{U}(p,q))$ in this paper based on [4] (see [39, Appendix A]). According to the persistence principle of Kudla (see [30]), the lift of $\theta_n(\Pi')$ of Π' on $\widetilde{\mathrm{G}}_n = \widetilde{\mathrm{U}}(n,n+1)$ is non-trivial.
In particular, if $n \geq 2$, both representations Π' and $\theta_n(\Pi')$ are sub-representations of ω_n (the metaplectic representation of $\mathrm{Sp}((\mathbb{C}^2 \otimes \mathbb{C}^{2n+1})_{\mathbb{R}})$) and the character $\Theta_{\theta_n(\Pi')}$ of $\theta_n(\Pi')$ is obtained via Chc^* (see Conjecture 1.2 and [49]). Using [4, Theorem 2.2] and [3, Theorem 0.9], we are able to give an explicit formula for the character $\Theta_{\theta_n(\Pi')}$ on $\widetilde{\mathrm{G}}_n^{\mathrm{reg}}$ (see [39, Theorem 6.5] and [39, Proposition 6.10]).
Notice that by a result of Li, the representation $\theta_n(\Pi')$ is irreducible and unitary. Moreover, we proved that the representations $\theta_n(\Pi')$ are not highest weight modules by proving that the wave front set $\mathrm{WF}(\theta_n(\Pi'))$ of $\theta_n(\Pi')$ (see [22]) does not satisfy $\mathrm{WF}(\theta_n(\Pi'))^2 = \{0\}$ (see [47] and [12]). So, the value of $\Theta_{\theta_n(\Pi')}$ cannot be obtained directly using Enright's formula [13].

1.2. **Extension of Howe's duality to Lie superalgebras and supergroups.** In [24, Theorem 8], Howe proved a one-to-one correspondence between some representations of a classical complex group (i.e. $\mathrm{G} = \mathrm{GL}(\mathrm{V})$, $\mathrm{O}(\mathrm{V})$ and $\mathrm{Sp}(\mathrm{V})$ where V is a finite dimensional vector space over $\mathbb{C}$) and a finite dimensional complex Lie superalgebra $\mathfrak{g}' = \mathfrak{g}'_{\bar{0}} \oplus \mathfrak{g}'_{\bar{1}}$ (see [51] for more details about Lie superalgebras and [8, Chapter 5] where the authors gave a detailed version of this duality using the language of super vector spaces and superalgebras).
More precisely, the pairs $(\mathrm{G},\mathfrak{g}')$ considered in his paper are the following:

- $(\mathrm{GL}(k,\mathbb{C}), \mathfrak{gl}(n|m,\mathbb{C}))$,
- $(\mathrm{Sp}(2k,\mathbb{C}), \mathfrak{osp}(2n|2m,\mathbb{C}))$,
- $(\mathrm{O}(k,\mathbb{C}), \mathfrak{spo}(2n|2m,\mathbb{C}))$.

The link between G and $\mathfrak{g}'$ is that $\mathfrak{g} = \mathrm{Lie}(\mathrm{G})$ and $\mathfrak{g}'$ form a dual pair in a certain orthosymplectic Lie superalgebra $\mathfrak{spo}(\mathrm{V}_{\bar{0}},\mathrm{V}_{\bar{1}})$, where both $\mathrm{V}_{\bar{0}}$ and $\mathrm{V}_{\bar{1}}$ are even dimensional complex vector spaces over $\mathbb{C}$ (see [8, Section 5.2]).
In [24], Howe constructed an action of $\mathfrak{spo}(\mathrm{V}_{\bar{0}},\mathrm{V}_{\bar{1}})$ on a space $\mathrm{S} = \mathrm{S}_{\bar{0}} \oplus \mathrm{S}_{\bar{1}}$; the space S is the supersymmetric algebra $\mathrm{SS}(\mathrm{U})$ on $\mathrm{U} = \mathrm{U}_{\bar{0}} \oplus \mathrm{U}_{\bar{1}}$ (see [8, Section 5.1.1]), where $\mathrm{U}_{\bar{0}}$ and $\mathrm{U}_{\bar{1}}$ are maximal isotropic subspaces of $\mathrm{V}_{\bar{0}}$ and $\mathrm{V}_{\bar{1}}$ (in particular, $\mathrm{V}_{\bar{0}} = \mathrm{U}_{\bar{0}} \oplus \mathrm{U}^*_{\bar{0}}$ and $\mathrm{V}_{\bar{1}} = \mathrm{U}_{\bar{1}} \oplus \mathrm{U}^*_{\bar{1}}$). Note that $\mathrm{SS}(\mathrm{U}) \cong \mathrm{S}(\mathrm{U}_{\bar{0}}) \times \Lambda(\mathrm{U}_{\bar{1}})$. The action of $\mathfrak{spo}(\mathrm{V}_{\bar{0}},\mathrm{V}_{\bar{1}})_{\bar{0}} \cong \mathfrak{sp}(\mathrm{V}_{\bar{0}}) \oplus \mathfrak{o}(\mathrm{V}_{\bar{1}})$ on $\mathrm{S} \cong \mathrm{S}(\mathrm{U}_{\bar{0}}) \otimes \Lambda(\mathrm{U}_{\bar{1}})$ is very special. Indeed, as mentioned in [24, Page 548], the corresponding action of $\mathfrak{o}(\mathrm{V}_{\bar{1}})$ on $\Lambda(\mathrm{U}_{\bar{1}})$ can be exponentiated to a group action (and the corresponding group is not $\mathrm{SO}(\mathrm{V}_{\bar{1}})$ but $\mathrm{Spin}(\mathrm{V}_{\bar{1}})$ (see [36])): this is the construction of the spinorial representation. Similarly, the space $\mathrm{S}(\mathrm{U}_{\bar{0}})$ can be embedded in a larger space with an action of $\mathfrak{sp}(\mathrm{V}_{\bar{0}})$ which can be exponentiated to a group action: this is the construction of the metaplectic representation of $\mathrm{Sp}(\mathrm{V}_{\bar{0}})$ (the space $\mathrm{V}_{\bar{0}}$ being complex, we don't need to consider a double cover because $\mathrm{Sp}(\mathrm{V}_{\bar{0}})$ is simply connected).
In Theorem [24, Theorem 8], Howe proved that

$$\mathrm{SS}(\mathrm{U}) = \bigoplus_{(\lambda,\mathrm{E}_\lambda)\in\mathrm{G}_d} \mathrm{E}_\lambda \otimes \mathrm{F}_\lambda \,, \tag{5}$$

where $(\lambda,\mathrm{E}_\lambda)$ is an irreducible finite-dimensional G-module, F_λ is an irreducible $\mathbb{Z}_2$-graded $\mathfrak{g}'$-module and where G_d is the set of finite dimensional irreducible modules of G such that $\mathrm{Hom}_{\mathrm{G}}(\mathrm{E}_\lambda, \mathrm{SS}(\mathrm{U})) \neq \{0\}$.

As explained previously, the link between G and $\mathfrak{g}'$ is that $(\mathfrak{g},\mathfrak{g}')$ is a dual pair in $\mathfrak{spo}(\mathrm{V}_{\bar{0}},\mathrm{V}_{\bar{1}})$. A natural question is the following: can we extend the previous duality to a general dual pair in $\mathfrak{spo}(\mathrm{V}_{\bar{0}},\mathrm{V}_{\bar{1}})$ (or in the corresponding supergroup $(\mathrm{Sp}(\mathrm{V}_{\bar{0}}) \times \mathrm{O}(\mathrm{V}_{\bar{1}}), \mathfrak{spo}(\mathrm{V}_{\bar{0}},\mathrm{V}_{\bar{1}}))$). With

Hadi Salmasian, we recently obtained a full classification of irreducible reductive dual pairs in $\mathfrak{spo}(V)$.

Dual pairs in an orthosymplectic Lie superalgebra (joint with Hadi Salmasian), In Preparation

In this paper, we classified the dual pairs in the orthosymplectic Lie superalgebra $\mathfrak{spo}(V)$, where $V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$. As in [34, Section 6] (see also [33, Lecture 5] or [42]), we first introduce the notion of reductivity and irreducibility for a dual pair in $\mathfrak{spo}(V)$ and prove by adapting the techniques of [34] that every reductive dual pair is a direct sum of irreducible pairs. Then by extending the ideas of [42] to our situation, we get a full classification of dual pairs in $\mathfrak{spo}(V)$. For example, in the complex orthosymplectic Lie superalgebra, every irreducible reductive dual pair $(\mathfrak{g}, \mathfrak{g}')$ is isomorphic to one of the following pairs:

- $(\mathfrak{spo}(2a|b, \mathbb{C}), \mathfrak{osp}(p|2q, \mathbb{C})) \subseteq \mathfrak{spo}(2(ap + bq)|4aq + bp, \mathbb{C})$,
- $(\tilde{\mathfrak{p}}(a, \mathbb{C}), \mathfrak{p}(p, \mathbb{C})) \subseteq \mathfrak{spo}(2ap|2ap, \mathbb{C})$,
- $(\mathfrak{gl}(a|b, \mathbb{C}), \mathfrak{gl}(p|q, \mathbb{C})) \subseteq \mathfrak{spo}(2(ap + bq)|2(aq + bp), \mathbb{C})$,
- $(\tilde{\mathfrak{q}}(a, \mathbb{C}), \mathfrak{q}(p, \mathbb{C})) \subseteq \mathfrak{spo}(2ap|2ap, \mathbb{C})$.

As a consequence, we get the classification of dual pairs in the orthosymplectic Lie supergroup $(\mathrm{Sp}(V_{\bar{0}}) \times \mathrm{O}(V_{\bar{1}}), \mathfrak{spo}(V_{\bar{0}}, V_{\bar{1}}))$. As in the symplectic case, those pairs will be the building blocks of the extension of Howe duality for Lie superalgebras/supergroups.

As mentioned before, the action of $\mathfrak{so}(V_{\bar{1}})$ on $\Lambda(U_{\bar{1}})$ is the infinitesimal version of the spinorial representation of $\mathrm{O}(V_{\bar{1}})$ (or its double cover $\mathrm{Pin}(V_{\bar{1}})$). Before intending to extend Howe's duality to the whole orthosymplectic group, it was natural to make sure that a similar phenomenon held for dual pairs in $\mathrm{Pin}(V_{\bar{1}})$. This was the motivation of the following work.

Dual pairs in the Pin-group and duality for the corresponding Spinorial representation (with C. Guérin and G. Liu, Algebras and Representation Theory, 2021)

Let E be a vector space over $\mathbb{K} = \mathbb{R}$ endowed with a non-degenerate, symmetric bilinear form b. The definition of irreducibility and reductivity for a dual pair in $\mathrm{O}(E, b)$ is similar to the one in the symplectic group (such dual pairs had been classified, see [52]). We denote by $\mathrm{Pin}(E, b)$ the Pin-group (it's a non-trivial two-fold cover of $\mathrm{O}(E, b)$ (see [36]) and by $\pi : \mathrm{Pin}(E, b) \to \mathrm{O}(E, b)$ the corresponding covering map. As pointed out by Slupinski in [54], one of the difference with the symplectic case is that the preimages in $\mathrm{Pin}(E, b)$ of a dual pair in $\mathrm{O}(E, b)$ do not necessarily commute in $\mathrm{Pin}(E, b)$. In [15, Section 3], using Slupinski's method in [54], we first answered the following question: For which irreducible reductive dual pair (G, G') in $\mathrm{O}(E, b)$ the preimages $(\widetilde{G}, \widetilde{G'})$ form a dual pair in $\mathrm{Pin}(E, b)$?

Let (Π, V_{Π}) be the spinorial representation of $\mathrm{Pin}(E, b)$ (see [36, Section 3]) and fix an irreducible reductive dual pair (G, G') in $\mathrm{O}(E, b)$ such that $(\widetilde{G}, \widetilde{G'})$ is a dual pair in $\mathrm{Pin}(E, b)$. As a representation of $\widetilde{G}$, we get the following decomposition:

$$V_{\Pi} = \bigoplus_{(\lambda, V_{\lambda}) \in \widetilde{G}_{\Pi}} m_{\lambda} V_{\lambda} = \bigoplus_{(\lambda, V_{\lambda}) \in \widetilde{G}_{\Pi}} V(\lambda), \tag{6}$$

where $\widetilde{G}_{\Pi}$ is the set of finite dimensional irreducible representations (λ, V_{λ}) of $\widetilde{G}$ such that $\mathrm{Hom}_{\widetilde{G}}(V_{\lambda}, V_{\Pi}) \neq \{0\}$, m_{λ} is the multiplicity of λ and $V(\lambda) = \{T(V_{\lambda}), T \in \mathrm{Hom}_{\widetilde{G}}(V_{\lambda}, V_{\Pi})\}$ is the λ-isotypic component. Because $\widetilde{G'}$ commute with $\widetilde{G}$, we get that for every $\lambda \in \widetilde{G}_{\Pi}$

$$V(\lambda) = \lambda \otimes \lambda',$$

where λ' is a representation of $\widetilde{\mathrm{G}}'$. In [15, Section 4], we proved using [26] that the representations λ' are irreducible, i.e. the correspondence $\lambda \leftrightarrow \lambda'$ is one-to-one.

2. Ongoing and Future Projects

Proof of Conjecture 1.2 for $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p,q), \mathrm{U}(r,s))$, $p+q = r+s$, for tempered representations

According to Xue [56], we can except that the results in [40] can be extended to tempered representations. As before, fix (p,q) and let r and s vary under the assumption that $p+q = r+s$. Then for every genuine representation Π of $\widetilde{\mathrm{U}}(p,q)$, there exists a unique pair (r,s) such that $p+q = r+s$ and $\theta_{r,s}(\Pi) \neq 0$. Moreover, according to [44], if Π is tempered, the corresponding representation Π' is tempered. Harish-Chandra gave a parametrisation of tempered distributions (see [55, Theorem 8.6.1]). Then by using the results of Paul ([44] and [45]), the conjecture 1.2 can be proved for (Π, Π'). As in the case of discrete series representations, we expect that the diagram 3 will commute.
Recently, W. T. Gan constructed a map $\mathrm{R} : \mathscr{D}'(\widetilde{\mathrm{G}})^{\widetilde{\mathrm{G}}} \to \mathscr{D}'(\widetilde{\mathrm{G}}')^{\widetilde{\mathrm{G}}'}$ in the equal rank case (for local non-archimedian fields) such that $\mathrm{R}(\Theta_\Pi) = \Theta_{\Pi'}$ for every tempered representations $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ (see [14]). It would be interesting to understand the link between the map R of [14] and the map Chc^*.

Character of the lift of a discrete series representation

In [40], I proved Conjecture 1.2 for a dual pair pair of unitary groups in the equal rank case starting with a discrete series representation. The project is now to prove Conjecture 1.2 for a general dual pair $(\mathrm{G}, \mathrm{G}')$, with $\mathrm{rk}(\mathrm{G}) \leq \mathrm{rk}(\mathrm{G}')$, starting with a discrete series representation Π of $\widetilde{\mathrm{G}}$. What makes Conjecture 1.2 complicated in general is the fact that there are no obvious relations between the Cauchy–Harish-Chandra integral and the characters of the representations (Π, Π'), $\Pi' = \theta(\Pi)$ (under our assumptions, it follows from [31, Proposition 2.4] that $\Pi'_1 = \Pi' = \theta(\Pi)$). In [40, Equation 7]), we proved that for every $\Psi \in \mathscr{C}^\infty_c(\widetilde{\mathrm{G}})$

$$\Theta_{\Pi'}(\Psi) = d_\Pi \mathrm{tr}\left(\int_{\widetilde{\mathrm{K}}}\int_{\widetilde{\mathrm{G}}}\int_{\widetilde{\mathrm{G}}'} \overline{\Theta_{\Pi_\nu}(\tilde{k})}\overline{\Theta_\Pi(\tilde{g})}\Psi(\tilde{g}')\omega(\tilde{k}\tilde{g}\tilde{g}')d\tilde{g}'d\tilde{g}d\tilde{k}\right),$$

where ν is the highest weight of the lowest-$\widetilde{\mathrm{K}}$-type of Π. In particular, by using Weyl's integration formula, we get:
(7)

$$\Theta_{\Pi'}(\Psi) = \sum_{i=1}^{n} \frac{d_\Pi}{|\mathscr{W}(\mathrm{H}_i)|}\int_{\widetilde{\mathrm{H}}_i} \overline{\Theta_\Pi(\tilde{h}_i)}|\Delta(\tilde{h}_i)|^2 \left(\int_{\widetilde{\mathrm{K}}}\int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{H}}_i}\int_{\widetilde{\mathrm{G}}'} \overline{\Theta_{\Pi_\nu}(\tilde{k})}\Theta(\tilde{k}\tilde{g}\tilde{h}_i\tilde{g}^{-1}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}'\overline{d\tilde{g}}d\tilde{k}\right)d\tilde{h}_i\,,$$

where $\mathrm{H}_1, \ldots, \mathrm{H}_n$ is a maximal set of Cartan subgroups of G ($n = \mathrm{rk}(\mathrm{G})$). Using (2),

$$\mathrm{Chc}^*(\Theta_\Pi)(\Psi) = \sum_{i=1}^{n} \frac{d_\Pi}{|\mathscr{W}(\mathrm{H}_i)|}\int_{\widetilde{\mathrm{H}}_i} \Theta_\Pi(\tilde{h}_i)|\Delta(\tilde{h}_i)|^2 \mathrm{Chc}_{\tilde{h}_i}(\Psi)d\tilde{h}_i \tag{8}$$

and in particular, we want to prove that

$$\sum_{i=1}^{n} \frac{1}{|\mathscr{W}(\mathrm{H}_i)|}\int_{\widetilde{\mathrm{H}}_i} \Theta_\Pi(\tilde{h}_i)|\Delta(\tilde{h}_i)|^2 \mathrm{Chc}_{\tilde{h}_i}(\Psi)d\tilde{h}_i \tag{9}$$

$$= d_\Pi \sum_{i=1}^{n} \frac{1}{|\mathscr{W}(\mathrm{H}_i)|} \int_{\widetilde{\mathrm{H}}_i} \overline{\Theta_\Pi(\tilde{h}_i)} |\Delta(\tilde{h}_i)|^2 \left(\int_{\widetilde{\mathrm{K}}} \int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{H}}_i} \int_{\widetilde{\mathrm{G}}'} \overline{\Theta_{\Pi_\nu}(\tilde{k})} \Theta(\tilde{k}\tilde{g}\tilde{h}_i\tilde{g}^{-1}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}'\overline{d\tilde{g}}d\tilde{k} \right) d\tilde{h}_i \,.$$

by working directly on the term

$$\int_{\widetilde{\mathrm{K}}} \int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{H}}_i} \int_{\widetilde{\mathrm{G}}'} \overline{\Theta_{\Pi_\nu}(\tilde{k})} \Theta(\tilde{k}\tilde{g}\tilde{h}_i\tilde{g}^{-1}\tilde{g}')\Psi(\tilde{g}')d\tilde{g}'\overline{d\tilde{g}}d\tilde{k} \,.$$

Howe duality for the dual pairs $(\mathfrak{spo}(m|n, \mathbb{C}), \mathfrak{osp}(s|t, \mathbb{C}))$ and $(\tilde{\mathfrak{p}}(n), \mathfrak{p}(m))$ (joint work with Hadi Salmasian)

In [41], we got a classification of irreducible reductive dual pairs $(\mathfrak{g}, \mathfrak{g}')$ in $\mathfrak{spo}(\mathrm{V})$, where $\mathrm{V} = \mathrm{V}_{\bar{0}} \oplus \mathrm{V}_{\bar{1}}$ is either real or complex. The next step is to look at the joint action of $\mathfrak{g}$ and $\mathfrak{g}'$ on the corresponding supersymmetric algebra SS(U) as in Section 1.2. A complete decomposition has been obtained for $(\mathfrak{gl}(m|n, \mathbb{C}), \mathfrak{gl}(s|t, \mathbb{C}))$ in [7] and for $(\tilde{\mathfrak{q}}(n), \mathfrak{q}(m))$ in [8]. In this project, we focus our attention on the dual pair $(\mathfrak{spo}(m|n, \mathbb{C}), \mathfrak{osp}(s|t, \mathbb{C}))$ and $(\tilde{\mathfrak{q}}(n), \mathfrak{q}(m))$. The corresponding actions of the Lie superalgebras we are working with on SS(U) are not completely reducible in general and in particular we cannot expect a decomposition as in (5). This was something expected because of how the duality is obtained in the symplectic case (as in Section 1.1). For the pair $(\tilde{\mathfrak{p}}(n), \mathfrak{p}(m))$, some results concerning the $\mathfrak{p}(n)$-invariants on the superalgebra SS(U) has been obtained recently in [9] but the question of Howe duality hasn't been treated (similar results had been obtained by Sergeev in [53]).

Duality for the Spinor-Oscillator representation

With the classification of irreducible reductive dual pairs in $\mathscr{G} = (\mathrm{Sp}(\mathrm{V}_{\bar{0}}) \times \mathrm{O}(\mathrm{V}_{\bar{1}}), \mathfrak{spo}(\mathrm{V}_{\bar{0}}, \mathrm{V}_{\bar{1}}))$ known (see [41]), the question of a general duality as in Section 1.1 arises naturally. In this case, we assume that $\mathscr{G} = (\mathrm{Sp}(\mathrm{V}_{\bar{0}}) \times \mathrm{O}(\mathrm{V}_{\bar{1}}), \mathfrak{spo}(\mathrm{V}))$ is a real Lie supergroup and let $\widetilde{\mathscr{G}}$ be the Lie supergroup $(\widetilde{\mathrm{Sp}}(\mathrm{V}_{\bar{0}}) \times \mathrm{Pin}(\mathrm{V}_{\bar{1}}), \mathfrak{spo}(\mathrm{V}))$. Using [40], we can answer the following question: for which dual pair $((\mathrm{G}, \mathfrak{g}), (\mathrm{G}', \mathfrak{g}'))$ in $\mathscr{G}$ their preimages $\left((\widetilde{\mathrm{G}}, \mathfrak{g}), (\widetilde{\mathrm{G}}', \mathfrak{g}')\right)$ form a dual pair in $\widetilde{\mathscr{G}}$? For the dual pairs we get in $\widetilde{\mathscr{G}}$, we can try to understand "the decomposition" of the spinor-oscillator representation $(\omega, \mathscr{H} = \mathscr{H}_{\bar{0}} \oplus \mathscr{H}_{\bar{1}})$ (see [27] for the extension in the real case). As for the duality in the symplectic case, the decomposition of $\mathscr{H}$ will not be a direct sum of irreducible $(\mathrm{G}, \mathfrak{g})$-modules. The first step is to understand the right category to work with to approach this duality. Note that the globalisation of Harish-Chandra supermodules had been studied by Alldridge in [1]. Moreover, the notion of unitary representation had been introduced in [6] and studied by many authors (one can check for example the article of Neeb and Salmasian [43]).
Finally, we can mention a first work on this subject, for the pair $(\mathrm{O}(p, q, \mathbb{R}), \widetilde{\mathrm{Osp}}(2, 2, \mathbb{R}))$, done by Howe and Lu in [35]. Even in this special, the results need to be completed.

References

[1] Alexander Alldridge. Fréchet globalisations of Harish-Chandra supermodules. *Int. Math. Res. Not. IMRN*, (17):5182–5232, 2017.

[2] Anne-Marie Aubert and Tomasz Przebinda. A reverse engineering approach to the Weil representation. *Cent. Eur. J. Math.*, 12(10):1500–1585, 2014.

[3] Florent Bernon and Tomasz Przebinda. Normalization of the Cauchy Harish-Chandra integral. *J. Lie Theory*, 21(3):615–702, 2011.

[4] Florent Bernon and Tomasz Przebinda. The Cauchy Harish-Chandra integral and the invariant eigendistributions. *Int. Math. Res. Not. IMRN*, (14):3818–3862, 2014.

[5] Abderrazak Bouaziz. Intégrales orbitales sur les groupes de Lie réductifs. *Ann. Sci. École Norm. Sup. (4)*, 27(5):573–609, 1994.
[6] C. Carmeli, G. Cassinelli, A. Toigo, and V. S. Varadarajan. Unitary representations of super Lie groups and applications to the classification and multiplet structure of super particles. *Comm. Math. Phys.*, 263(1):217–258, 2006.
[7] Shun-Jen Cheng and Weiqiang Wang. Howe duality for Lie superalgebras. *Compositio Math.*, 128(1):55–94, 2001.
[8] Shun-Jen Cheng and Weiqiang Wang. *Dualities and representations of Lie superalgebras*, volume 144 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.
[9] Nicholas Davidson, Jonathan R. Kujawa, and Robert Muth. Howe duality of type p. *arXiv:2109.03984*, 2021.
[10] Michel Duflo and Michèle Vergne. Orbites coadjointes et cohomologie équivariante. In *The orbit method in representation theory (Copenhagen, 1988)*, volume 82 of *Progr. Math.*, pages 11–60. Birkhäuser Boston, Boston, MA, 1990.
[11] Thomas Enright, Roger Howe, and Nolan Wallach. A classification of unitary highest weight modules. In *Representation theory of reductive groups (Park City, Utah, 1982)*, volume 40 of *Progr. Math.*, pages 97–143. Birkhäuser Boston, Boston, MA, 1983.
[12] Thomas Enright, Roger Howe, and Nolan Wallach. A classification of unitary highest weight modules. In *Representation theory of reductive groups (Park City, Utah, 1982)*, volume 40 of *Progr. Math.*, pages 97–143. Birkhäuser Boston, Boston, MA, 1983.
[13] Thomas J. Enright. Analogues of Kostant's u-cohomology formulas for unitary highest weight modules. *J. Reine Angew. Math.*, 392:27–36, 1988.
[14] Wee Teck Gan. Transfer of characters under the howe duality correspondence. *In Preparation*, 2020.
[15] Clément Guérin, Gang Liu, and Allan Merino. Dual Pairs in the Pin-Group and Duality for the Corresponding Spinorial Representation. *Algebr. Represent. Theory*, 24(6):1625–1640, 2021.
[16] Harish-Chandra. Representations of semisimple Lie groups. III. Characters. *Proc. Nat. Acad. Sci. U.S.A.*, 37:366–369, 1951.
[17] Harish-Chandra. Representations of semisimple Lie groups on a Banch space. *Proc. Nat. Acad. Sci. U.S.A.*, 37:170–173, 1951.
[18] Harish-Chandra. Discrete series for semisimple Lie groups. I. Construction of invariant eigendistributions. *Acta Math.*, 113:241–318, 1965.
[19] Harish-Chandra. Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119:457–508, 1965.
[20] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. *Acta Math.*, 116:1–111, 1966.
[21] Henryk Hecht. The characters of some representations of Harish-Chandra. *Math. Ann.*, 219(3):213–226, 1976.
[22] Roger Howe. Wave front sets of representations of Lie groups. In *Automorphic forms, representation theory and arithmetic (Bombay, 1979)*, volume 10 of *Tata Inst. Fund. Res. Studies in Math.*, pages 117–140. Tata Institute of Fundamental Research, Bombay, 1981.
[23] Roger Howe. The oscillator semigroup. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 61–132. Amer. Math. Soc., Providence, RI, 1988.
[24] Roger Howe. Remarks on classical invariant theory. *Trans. Amer. Math. Soc.*, 313(2):539–570, 1989.
[25] Roger Howe. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535–552, 1989.
[26] Roger Howe. Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond. In *The Schur lectures (1992) (Tel Aviv)*, volume 8 of *Israel Math. Conf. Proc.*, pages 1–182. Bar-Ilan Univ., Ramat Gan, 1995.
[27] A. Huckleberry, A. Püttmann, and M. R. Zirnbauer. Haar expectations of ratios of random characteristic polynomials. *Complex Anal. Synerg.*, 2(1):Paper No. 1, 73, 2016.
[28] A. A. Kirillov. Characters of unitary representations of Lie groups. Reduction theorems. *Funkcional. Anal. i Priložen.*, 3(1):36–47, 1969.
[29] Anthony W. Knapp. *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.
[30] Stephen S. Kudla. On the local theta-correspondence. *Invent. Math.*, 83(2):229–255, 1986.
[31] Jian-Shu Li. Singular unitary representations of classical groups. *Invent. Math.*, 97(2):237–255, 1989.

[32] Jian-Shu Li. Theta lifting for unitary representations with nonzero cohomology. *Duke Math. J.*, 61(3):913–937, 1990.

[33] Jian-Shu Li. Minimal representations & reductive dual pairs. In *Representation theory of Lie groups (Park City, UT, 1998)*, volume 8 of *IAS/Park City Math. Ser.*, pages 293–340. Amer. Math. Soc., Providence, RI, 2000.

[34] Dan Lu and Roger Howe. The dual pair $(O_{p,q}, \tilde{OSp}_{2,2})$ and Zuckerman translation. In *Representation theory and mathematical physics*, volume 557 of *Contemp. Math.*, pages 183–213. Amer. Math. Soc., Providence, RI, 2011.

[35] Dan Lu and Roger Howe. The dual pair $(O_{p,q}, \tilde{OSp}_{2,2})$ and Zuckerman translation. In *Representation theory and mathematical physics*, volume 557 of *Contemp. Math.*, pages 183–213. Amer. Math. Soc., Providence, RI, 2011.

[36] Eckhard Meinrenken. *Clifford algebras and Lie theory*, volume 58 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer, Heidelberg, 2013.

[37] Allan Merino. Characters of some unitary highest weight representations via the theta correspondence. *J. Funct. Anal.*, 279(8):108698, 70, 2020.

[38] Allan Merino. Transfer of characters in the theta correspondence with one compact member. *J. Lie Theory*, 30(4):997–1026, 2020.

[39] Allan Merino. Characters of irreducible unitary representations of $\mathrm{U}(n, n+1)$ via double lifting from $\mathrm{U}(1)$. *Represent. Theory*, 26:325–369, 2022.

[40] Allan Merino. Transfer of characters for discrete series representations of the unitary groups in the equal rank case via the cauchy-harish-chandra integral. *International Mathematics Research Notices*, 2022.

[41] Allan Merino and Hadi Salmasian. Dual pairs in an orthosymplectic lie superalgebra. *In Preparation*, 2021.

[42] Colette Mœ glin, Marie-France Vignéras, and Jean-Loup Waldspurger. *Correspondances de Howe sur un corps p-adique*, volume 1291 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[43] Karl-Hermann Neeb and Hadi Salmasian. Lie supergroups, unitary representations, and invariant cones. In *Supersymmetry in mathematics and physics*, volume 2027 of *Lecture Notes in Math.*, pages 195–239. Springer, Heidelberg, 2011.

[44] Annegret Paul. Howe correspondence for real unitary groups. *J. Funct. Anal.*, 159(2):384–431, 1998.

[45] Annegret Paul. First occurrence for the dual pairs $(\mathrm{U}(p, q), \mathrm{U}(r, s))$. *Canad. J. Math.*, 51(3):636–657, 1999.

[46] Tomasz Przebinda. Characters, dual pairs, and unipotent representations. *J. Funct. Anal.*, 98(1):59–96, 1991.

[47] Tomasz Przebinda. Characters, dual pairs, and unitary representations. *Duke Math. J.*, 69(3):547–592, 1993.

[48] Tomasz Przebinda. A Cauchy Harish-Chandra integral, for a real reductive dual pair. *Invent. Math.*, 141(2):299–363, 2000.

[49] Tomasz Przebinda. The character and the wave front set correspondence in the stable range. *J. Funct. Anal.*, 274(5):1284–1305, 2018.

[50] Wulf Rossmann. *Lie groups*, volume 5 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. An introduction through linear groups.

[51] Manfred Scheunert. *The theory of Lie superalgebras*, volume 716 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979. An introduction.

[52] Wilfried Schmid. On the characters of the discrete series. The Hermitian symmetric case. *Invent. Math.*, 30(1):47–144, 1975.

[53] Alexander Sergeev. An analog of the classical invariant theory for Lie superalgebras. I, II. *Michigan Math. J.*, 49(1):113–146, 147–168, 2001.

[54] M. J. Slupinski. Dual pairs in $\mathrm{Pin}(p, q)$ and Howe correspondences for the spin representation. *J. Algebra*, 202(2):512–540, 1998.

[55] Nolan R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.

[56] Hang Xue. Arithmetic theta lifts and the arithmetic Gan-Gross-Prasad conjecture for unitary groups. *Duke Math. J.*, 168(1):127–185, 2019.