MA6292: Character of quasi-simple representations

Allan Merino
National University of Singapore
matafm@nus.edu.sg

Year 2019/2020 - Semester 2

Contents

Chapter 1

Introduction

The goal of this course is to understand Harish-Chandra's theory of characters of quasi-simple representations. If we want to summarise the main result of this course, it can be stated as follow:

Theorem 1.0.1. *Let* G *be a real reductive Lie group and* $(\Pi, \mathscr{H})$ *an irreducible quasi-simple representation of* G. *For every* $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, *the operator* $\Pi(\Psi)$ *is of trace class and the corresponding map:*

$$\Theta_\Pi : \mathscr{C}_c^\infty(\mathrm{G}) \ni \Psi \to \mathrm{tr}(\Pi(\Psi)) \in \mathbb{C}$$

is a distribution (in the sense of Laurent Schwartz). Moreover, there exists a locally integrable function F_Π *on* G *which is analytic on the set of regular points* G' *of* G *such that* $\Theta_\Pi = T_{F_\Pi}$, *i.e.*

$$\Theta_\Pi(\Psi) = \mathrm{tr} \int_{\mathrm{G}} \Psi(g)\Pi(g)dg = \int_{\mathrm{G}} \Psi(g)F_\Pi(g)dg \qquad (\Psi \in \mathscr{C}_c^\infty(\mathrm{G})).$$

Finally, two irreducible quasi-simple representations $(\Pi_1, \mathscr{H}_1)$ *and* $(\Pi_2, \mathscr{H}_2)$ *are infinitesimally equivalent if and only if* $\Theta_{\Pi_1} = \Theta_{\Pi_2}$.

Let's say few words about what is contained in those notes. In this text

1. G will be a reductive real Lie group (most of the time assumed to be connected),
2. $\mathfrak{g}_0$ the Lie algebra of G, $\mathfrak{g}$ its complexification,
3. θ Cartan involution on $\mathfrak{g}$ such that $\theta(\mathfrak{g}_0) = \mathfrak{g}_0$, $\mathfrak{k}$ (resp. $\mathfrak{p}$) eigenspace corresponding to the eigenvalue 1 (resp. -1),
4. $c_0 = \mathrm{Z}(\mathfrak{k}_0)$, D analytic subgroup of c_0, K Lie group such that $\mathrm{Lie}(\mathrm{K}) = \mathfrak{k}_0$ (not compact in general)
5. $\mathrm{K}^* = \mathrm{K} / \mathrm{Z}(\mathrm{G}) \cap D$ (compact).

The course is divided in five main parts:

In chapter 2, we start by recalling the main results of the representation of compact Lie groups. In particular, we explain why all the irreducible representations are finite dimensional, and in particular that all those representations are parametrised by a linear form on a Cartan subalgebra $\mathfrak{t}$ on $\mathfrak{g}$. The character of such representations can be computed using this linear form: that's the Weyl Character formula (see (2.6)).

In chapter 3, we first introduce and prove some important properties of the Harish-Chandra space. One of the difference between finite and infinite dimensional representations $(\Pi, \mathscr{H})$ is that the Lie algebra $\mathfrak{g}_0$ of G does not act on $\mathscr{H}$ in general. Garding introduced a dense subspace $\mathrm{Gar}(\Pi, \mathscr{H})$ of $\mathscr{H}$ but Harish-Chandra pointed out that the closure $\overline{\mathscr{F}}$ of a $\mathfrak{g}_0$-invariant subspace $\mathscr{F}$ of $\mathrm{Gar}(\Pi, \mathscr{H})$ is not necessarily G-invariant. We prove in this chapter that the space of analytic vectors $\mathrm{Har}(\Pi, \mathscr{H})$ (or Harish-Chandra space, see Corollary 3.3.10) solve the previous issue.

Definition 1.0.2. 1. A representation $(\Pi, \mathscr{H})$ is said to be permissible if the restriction of $\Pi(z)$ acts a scalar multiple of the unit operator $(z \in \mathrm{Z}(\mathrm{G}) \cap \mathrm{D})$.

2. A representation $(\Pi, \mathscr{H})$ is said to be quasi-simple if Π is permissible and if here exists an homomorphism χ of $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ into $\mathbb{C}$ such that $d\Pi_{\mathrm{G}}(z)\Psi = \chi(z)\Psi$ for every $z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ and $\Psi \in \mathrm{Gar}(\Pi, \mathscr{H})$.

In this chapter, we prove that if $(\Pi, \mathscr{H})$ is permissible, the Harish-Chandra space $\mathrm{Har}(\Pi, \mathscr{H})$ is dense in $\mathscr{H}$. Moreover, if $(\Pi, \mathscr{H})$ is quasi-simple, the K-isotypic components are finite dimensional.

In Chapter 4, we first prove that the trace of the operators $\Pi(\Psi), \Psi \in \mathscr{C}^\infty_c(\mathrm{G})$, exists. After, we prove that the corresponding map on $\mathscr{C}^\infty_c(\mathrm{G})$ is an invariant distribution and that two representations Π_1 and Π_2 are infinitesimally equivalent if and only if the corresponding global characters Θ_{Π_1} and Θ_{Π_2} are equals. We finish the chapter by proving that if both representations are unitary, the notion of infinitesimally equivalence and equivalence coincide.
In Chapter 5, after recalling and proving properties of differential operators on Lie groups, we prove the following theorem:

Theorem 1.0.3. *Let Ω be a completely invariant open set in G and T a distribution on Ω. We assume that*

1. *T is invariant,*

2. *There exists an ideal $\mathscr{U}$ in $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ such that $\dim_{\mathbb{C}} \mathrm{Z}(\mathscr{U}(\mathfrak{g}))/\mathscr{U} < \infty$ and $\eta(u)T = 0$ for $u \in \mathscr{U}$.*

Then, T is given by a function $F_T \in \mathrm{L}^1_{loc}(\Omega)$ which is analytic on $\Omega' = \Omega \cap \mathrm{G}'$.

This theorem does not depend of any representations of G, but by definition of a quasi-simple representation $(\Pi, \mathscr{H})$, we get that $z\Theta_\Pi = \chi_\Pi(z)\Theta_\Pi, z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))$, where χ_Π is a character of the center of $\mathscr{U}(\mathfrak{g})$. In particular, it implies that Θ_Π coincides with a locally integrable function on G analytic on G′.

In Chapter 6, we recall some basic results about Fourier transform of orbits and a conjecture of Kirillov concerning the character of an irreducible unitary representation of a Lie group. In particular, using a result of Harish-Chandra and a paper of Rossmann, we give an explicit formula for the Fourier transform of (some) orbits and prove that the Kirillov conjecture holds for discrete series representations. We finish this chapter by giving some open problems in character theory.

Chapter 2

Representation Theory of Compact Lie Groups and Weyl's Character formula

2.1 Basics about representations of compact groups

Throughout this section, G will be a compact topological group and μ will denote the normalised Haar measure on G, i.e. $dg(\mathrm{G}) = 1$. In general, we always have to precise if we are dealing with left or right measures, but because the groups we consider are compacts, they are unimodulars. Moreover, even if it is possible to work with locally convex vector spaces, we will only deal with Hilbert spaces (finite or infinite). Let's start with some notations and definitions.
A (continuous) representation of G is a pair $(\Pi, \mathscr{H})$ where $(\mathscr{H}, \langle \cdot, \cdot \rangle)$ is a complex Hilbert space and $\Pi : G \to \mathscr{L}(\mathscr{H})$ is a map satisfying:

1. Π is a group homomorphism, i.e. $\Pi(gh) = \Pi(g)\Pi(h), g, h \in \mathrm{G}$, and $\Pi(e) = \mathrm{Id}$,

2. For all $v \in \mathscr{H}$, the map $\mathrm{G} \ni g \to \Pi(g)(v) \in \mathscr{H}$ is continuous.

Here, $\mathscr{L}(\mathscr{H})$ is the set of bounded linear operators endowed with the weak topology (so necessarily continuous because $\mathscr{H}$ is an Hilbert space).
An invariant subspace of $(\Pi, \mathscr{H})$ is a closed subspace $\mathscr{H}_0$ of $\mathscr{H}$ which is invariant by the operators $\Pi(g), g \in \mathrm{G}$. The representation is called irreducible if there is no proper closed invariant subspaces.
The representation is said unitary if the operators $\Pi(g), g \in \mathrm{G}$, are all unitary with respect to the inner product $\langle \cdot, \cdot \rangle$, i.e.

$$\langle \Pi(g)u, \Pi(g)v \rangle = \langle u, v \rangle \qquad (u, v \in \mathscr{H}, g \in \mathrm{G}). \tag{2.1}$$

Remark 2.1.1. 1. For all $A \in \mathscr{L}(\mathscr{H})$, we denote by A^* the adjoint operator of A (existence and uniqueness is given by Riesz's Theorem). On $\mathscr{L}(\mathscr{H})$, we define the norm N given by:

$$N(A) = \sup_{\substack{v \in \mathscr{H} \\ \|v\|=1}} \|A(v)\|,$$

where $\|\cdot\|$ is the norm on $\mathscr{H}$ coming from the inner product $\langle\cdot,\cdot\rangle$. In general, we have $\|A\| = \|A^*\|$, and if A is unitary, we get $\|A\| = 1$.

2. The equality (2.1) is equivalent to the following equalities:

   - $\|\Pi(g)(u)\| = \|u\|, u \in \mathscr{H}, g \in \mathrm{G}$,
   - $\Pi(g)\Pi(g)^* = \Pi(g)^*\Pi(g) = \mathrm{Id}, g \in \mathrm{G}$,
   - $\langle \Pi(g)u, v\rangle = \left\langle u, \Pi(g)^{-1}v\right\rangle, u, v \in \mathscr{H}, g \in \mathrm{G}$.

One goal of this section is to prove that all the irreducible representations of a compact topological group are finite dimensional. Let's first state basics results about finite dimensional representations of G (not necessarily irreducible).

Theorem 2.1.2. *Let* $(\Pi, \mathscr{H})$ *be a finite dimensional representation of* G*.*

1. *There exists an hermitian inner product* $\langle\cdot,\cdot\rangle_0$ *on* $\mathscr{H}$ *such that* Π *is unitary,*
2. *If* $\mathscr{H}_0$ *is a closed* G*-invariant subspace of* $\mathscr{H}$*, then* $\mathscr{H}_0^\perp$ *is* G*-invariant,*
3. *The representation* Π *is the direct sum of irreducible representations of* G*.*

Proof. 1. We define $\langle\cdot,\cdot\rangle_0$ by:

$$\langle u, v\rangle_0 = \int_{\mathrm{G}} \langle \Pi(g)(u), \Pi(g)(v)\rangle \, d\mu(g) \qquad (u, v \in \mathscr{H}).$$

It's straightforward to prove that $\langle\cdot,\cdot\rangle_0$ is a G-invariant hermitian inner product on $\mathscr{H}$.

2. The space $\mathscr{H}_0^\perp$ is given by $\mathscr{H}_0^\perp = \{v \in \mathscr{H}_0, \langle v, u\rangle = 0\, (\forall u \in \mathscr{H}_0)\}$. For all $g \in \mathrm{G}$, $v^\perp \in \mathscr{H}_0^\perp$ and $v \in \mathscr{H}_0$, we get:

$$\langle \Pi(g)v^\perp, v\rangle = \left\langle v^\perp, \Pi(g^{-1})v\right\rangle = 0.$$

3. Obvious by induction on $\dim_{\mathbb{C}}(\mathscr{H})$.

□

Remark 2.1.3. The first point of Theorem 2.1.2 is still valid if the representation is infinite dimensional (and the proof is similar).

Let's now prove that every irreducible representation of G is finite dimensional. We first recall some well-known facts concerning compact operators on an Hilbert space.

Recall 2.1.4. Let $\mathscr{H}$ be a complex Hilbert space and $A \in \mathscr{L}(\mathscr{H})$. The operator A is said self-adjoint if $A = A^*$. Moreover, the operator A is said compact is the closure of $A(\{h \in \mathscr{H}, \|h\| \leq 1\})$ is compact in $\mathscr{H}$. We give some properties concerning compact operators. Let A be an operator in $\mathscr{L}(\mathscr{H})$ and let's denote by $\mathscr{C}(\mathscr{H})$ the set of compact operators on $\mathscr{H}$.

- For every $B \in \mathscr{C}(\mathscr{H})$, both AB and BA are in $\mathscr{C}(\mathscr{H})$.
- The operator A is compact if and only if A^* is compact.
- If there exists a sequence $(A_n)_{n\in\mathbb{N}}$ of operators in $\mathscr{C}(\mathscr{H})$ such that $\lim_{n\to+\infty} N(A - A_n) = 0$, then A is compact. In particular, $\mathscr{C}(\mathscr{H})$ is closed in $\mathscr{L}(\mathscr{H})$.

We now state the so-called spectal theorem for self-adjoint compact operators: Let A be a self-adjoint operator in $\mathscr{C}(\mathscr{H})$. Then, there exists an orthonormal basis $\{e_n\}$ of eigenvectors of $\mathscr{H}$ with real eigenvalues λ_n such that

$$A = \sum_n \lambda_n \langle e_n, \cdot \rangle e_n,$$

where the multiplicity of each non-zero eigenvalue is finite, and if $\mathscr{H}$ is infinite-dimensional, we have $\lim_{n\to\infty} \lambda_n = 0$.

Proposition 2.1.5. *Every (unitary) representation of a compact topological group possesses a finite dimensional G-invariant subspace. In particular, an irreducible representation of a compact group is finite dimensional.*

Proof. For every $v \in \mathscr{H}$, we define the operator K_v by:

$$K_v(w) = \int_G \langle w, \Pi(g)v \rangle \Pi(g)v d\mu(g) = \int_G P_{\Pi(g)v}(w) d\mu(g),$$

where the map $P_v : \mathscr{H} \ni w \to \langle w, v \rangle v \in \mathbb{C}$ is the projection on the space $\mathbb{C}v$. The operator K_v satisfy the following properties:

1. $K_v \neq 0$ if $v \neq 0$. Indeed, we have:

$$\langle K_v(v), v \rangle = \int_G \langle \Pi(g)v, v \rangle \overline{\langle \Pi(g)v, v \rangle} d\mu(g) = \int_G |\langle \Pi(g)v, v \rangle|^2 d\mu(g).$$

   Because the function $f : G \ni g \to | < \Pi(g)v, v > |^2 \in \mathbb{C}$ is continuous and that $f(e) = \|v\|^2 > 0$. In particular, there exists an open neighbourhood $\mathscr{U}$ of $e \in G$ such that $f(w) \geq \frac{\|v\|^2}{2}$ for all $w \in \mathscr{U}$. In particular,

$$\int_G |\langle \Pi(g)v, v \rangle|^2 d\mu(g) \geq \frac{\|v\|^2 \mu(\mathscr{U})}{2} > 0.$$

2. $K_v \in \mathscr{L}(\mathscr{H})$ and $N(K_v) \leq \|v\|^2$. Moreover, K_v is self-adjoint.

3. The operator K_v is an intertwining operator, i.e. $(\forall g \in \mathrm{G})$, $K_v \circ \Pi(g) = \Pi(g) \circ K_v$.

4. The operator K_v is compact.

Let's fix a non-zero vector $v \in \mathscr{H}$ and let K_v be the corresponding compact operator (which is a self-adjoint compact operator). Let λ be an eigenvalue of K_v and let $\mathscr{H}_\lambda$ be the corresponding eigenspace. Because K_v intertwines the representation Π, the space $\mathscr{H}_\lambda$ is G-invariant.

□

We now recall some famous integral relations that will be useful for us in the third section: they are known as Schur's orthogonality relations.

Proposition 2.1.6. *1. Let $(\Pi, \mathscr{H})$ be an irreducible representation of G. Then, for every $u_1, u_2, v_1, v_2 \in \mathscr{H}$, we get:*

$$\int_{\mathrm{G}} \langle \Pi(g)u_1, v_1\rangle \overline{\langle \Pi(g)u_2, v_2\rangle} d\mu(g) = \frac{\langle u_1, u_2\rangle \overline{\langle v_1, v_2\rangle}}{\dim_{\mathbb{C}}(V)}. \tag{2.2}$$

2. Let $(\Pi, \mathscr{H})$ and $(\Pi', \mathscr{H}')$ be inequivalent irreducible unitary representations of G (we denote by $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle'$ the hermitian inner product on $\mathscr{H}$ and $\mathscr{H}'$ respectively). Then, for all $u_1, u_2 \in \mathscr{H}$ and $u_1', u_2' \in \mathscr{H}'$, we get:

$$\int_{\mathrm{G}} \langle \Pi(g)u_1, u_2\rangle \overline{\left\langle \Pi'(g)u_1', u_2'\right\rangle} d\mu(g) = 0.$$

Proof. Let $L : \mathscr{H}' \to \mathscr{H}$ be a linear form and $\widetilde{L}$ defined by

$$\widetilde{L} = \int_{\mathrm{G}} \Pi(g) \circ L \circ \Pi'(g)^{-1} d\mu(g).$$

Obviously, for every $g \in \mathrm{G}$, we get $\Pi(g) \circ \widetilde{L} = \widetilde{L} \circ \Pi'(g)$. If the two representations are non-equivalent, according to Schur's lemma, we get $\widetilde{L} = 0$. Let L be the map defined by $L(w') =< w', u_1' > u_1$. We get:

$$\begin{aligned}
0 &= \left\langle \widetilde{L}(u_2'), u_2 \right\rangle = \left\langle \int_{\mathrm{G}} \Pi(g) \circ L \circ \Pi'(g) u_2' d\mu(g), u_2 \right\rangle = \int_{\mathrm{G}} \langle \Pi(g) \circ L \circ \Pi'(g) u_2', u_2\rangle \, d\mu(g) \\
&= \int_{\mathrm{G}} \left\langle \Pi(g) \left(\left\langle \Pi'(g)^{-1} u_2', u_1' \right\rangle u_1 \right), u_2 \right\rangle d\mu(g) = \int_{\mathrm{G}} \langle \Pi(g)u_1, u_2\rangle \left\langle \Pi'(g)^{-1} u_2', u_1' \right\rangle d\mu(g) \\
&= \int_{\mathrm{G}} \langle \Pi(g)u_1, u_2\rangle \overline{\left\langle \Pi'(g)u_1', u_2' \right\rangle} d\mu(g)
\end{aligned}$$

□

For a finite dimensional representation $(\Pi, \mathscr{H})$ of G, we associate the function $\Theta_\Pi : \mathrm{G} \to \mathbb{C}$ defined by:

$$\Theta_\Pi(g) = \mathrm{tr}(\Pi(g)) \qquad (g \in \mathrm{G}).$$

The function Θ_Π is called the character of Π. Obviously, by fixing an orthogonal basis $\{v_1, \ldots, v_n\}$ where $n = \dim_{\mathbb{C}}(\mathscr{H})$, we get that, for $g \in \mathrm{G}$:

$$\Theta_\Pi(g) = \sum_{i=1}^{n} \langle \Pi(g)v_i, v_i \rangle . \tag{2.3}$$

Then, by using Equation (2.2), we get:

$$\begin{aligned}\int_{\mathrm{G}} \Theta_\Pi(g)\overline{\Theta_\Pi(g)}dg &= \sum_{i=1}^{n}\sum_{j=1}^{n} \int_{\mathrm{G}} \langle \Pi(g)v_i, v_i \rangle \overline{\left\langle \Pi(g)v_j, v_j \right\rangle} d\mu(g) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\left\langle v_i, v_j \right\rangle \overline{\left\langle v_i, v_j \right\rangle}}{\dim_{\mathbb{C}}(\mathscr{H})} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\delta_{i,j}}{\dim_{\mathbb{C}}(\mathscr{H})} = 1.\end{aligned}$$

More generally, for two irreducible representations $(\Pi, \mathscr{H})$ and $(\Pi', \mathscr{H}')$ of G, we get:

$$\int_{\mathrm{G}} \Theta_\Pi(g)\overline{\Theta_{\Pi'}(g)}d\mu(g) = \begin{cases} 1 & \textbf{if } \Pi \sim \Pi' \\ 0 & \textbf{if } \Pi \neq \Pi' \end{cases}$$

2.2 Representation of complex semi-simple Lie algebras

We start this section by recalling simple definitions and properties of Lie algebras.
Let $(\mathfrak{g}, [\cdot,\cdot])$ be a finite dimensional Lie algebra over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. The Lie algebra $\mathfrak{g}$ is said simple if $\mathfrak{g}$ is nonabelian and $\mathfrak{g}$ has no proper nonzero ideals, and semisimple if $\mathfrak{g}$ has no nonzero solvable ideals. Obviously, the center of a semisimple Lie algebra is trivial.
We denote by $ad : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ the map given by $ad(X)(Y) = [X, Y]$ and by $\mathscr{K}$ the Killing form on $\mathfrak{g}$, i.e.

$$\mathscr{K} : \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \to \mathrm{tr}(ad(X) \circ ad(Y)) \in \mathbb{C}.$$

Theorem 2.2.1. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$*

1. *(Cartan) The Lie algebra $\mathfrak{g}$ is semisimple if and only if its corresponding Killing form is non-degenerate.*

2. *If $\mathfrak{g}$ is semisimple, then $ad(\mathfrak{g}) = Der(\mathfrak{g})$, where $Der(\mathfrak{g})$ corresponds to the set of derivations of $\mathfrak{g}$, i.e.*

$$Der(\mathfrak{g}) = \{D : \mathfrak{g} \to \mathfrak{g} \textit{ linear }, D([X, Y]) = [D(X), Y] + [X, D(Y)] \ (\forall X, Y \in \mathfrak{g})\} .$$

3. *Every finite dimensional representation of a semisimple Lie algebra is completely reducible.*

We now recall the abstract Jordan decomposition. Let $\mathfrak{g}$ be a complex semisimple Lie algebra. It is well-known that for all endomorphism $A \in \mathrm{End}(\mathfrak{g})$, there exists two endormorphisms B, C of $\mathrm{End}(\mathfrak{g})$ such that $A = B + C$, where B is semisimple, C is nilpotent and B and C commute.

Lemma 2.2.2. *For all $X \in \mathfrak{g}$, there exists two elements X_s, X_n in $\mathfrak{g}$ satisfying $[X_s, X_n] = 0$ and such that $ad(X) = ad(X_s) + ad(X_n)$. In particular, the decomposition $X = X_s + X_n$ is called the abstract Jordan decomposition of X and X_s (resp. X_n) the semisimple (resp. nilpotent) part of X.*

Proof. Let's first notice that the map $\mathfrak{g} \ni X \to ad(X) \in \mathrm{End}(\mathfrak{g})$ is 1-1. Indeed, if $ad(X_1) = ad(X_2)$, it means that $ad(X_1 - X_2) = 0$, i.e. $\mathbb{C}(X_1 - X_2)$ is a (solvable) ideal of $\mathfrak{g}$. Then, $\mathbb{C}(X_1 - X_2) = \{0\}$, i.e. $X_1 = X_2$.
Let X be an element of $\mathfrak{g}$. Then, using Theorem 2.2.1, there exists a derivation $D_X \in Der(\mathfrak{g})$ such that $ad(X) = D_X$. According to [20, Chapter 4.2, Lemma B], $Der(\mathfrak{g})$ contains the semisimple and nilpotent part of its Jordan decomposition. Then, there exists $D_1^X, D_2^X \in Der(\mathfrak{g})$ such that $D_X = D_1^X + D_2^X$ with D_1^X semisimple. Again, using Theorem 2.2.1, there exists X_s and X_n two elements of $\mathfrak{g}$ such that $D_1^X = ad(X_s)$ and $D_2^X = ad(X_n)$.

□

Let $\mathfrak{g}$ be a semisimple algebra. A toral algebra is a non-zero subalgebra consisting of semisimple elements. Those algebras exists and are abelian (see [20, Section 8.1]). A maximal toral algebra $\mathfrak{t}$ of $\mathfrak{g}$ is called a Cartan subalgebra.
We mention a last important result before recalling the concept of roots for a semisimple Lie algebra.

Theorem 2.2.3. *Let $\mathfrak{g}$ be a semisimple Lie algebra and X be a semisimple (resp. nilpotent) element of $\mathfrak{g}$. Then, for any finite dimensional representation (Π, V) of $\mathfrak{g}$, the endomorphism $\Pi(X)$ is semisimple (resp. nilpotent).*

For a proof of this result, one can check [20, Corollary 6.4].

2.2.1 Root system of a complex semisimple Lie algebra

Throughout this section, $\mathfrak{g}$ will be a complex semisimple Lie algebra and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$.
As explained before, the endomorphisms $ad(x)$ are diagonalizable for all $x \in \mathfrak{t}$ and because $\mathfrak{t}$ is abelian, those endomorphisms commute. In particular, we can diagonalize these endomorphisms simultaneously. In particular, the space $\mathfrak{g}$ is the direct sum of the subspaces $\mathfrak{g}_\alpha$,

eigenspace of the endomorphisms $ad(x)$, $x \in \mathfrak{t}$, corresponding to the "eigenvalue" $\alpha \in \mathfrak{t}^*$, i.e:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}, ad(h)(X) = \alpha(h)X, h \in \mathfrak{t}\}.$$

Obviously, because $\mathfrak{t}$ is abelian, we get that $\mathfrak{g}_0 \neq 0$. More precisely, we have $\mathfrak{g}_0 = \mathscr{C}_\mathfrak{g}(\mathfrak{t})$.
We denote by Φ the subset of $\mathfrak{t}^*$ given by:

$$\Phi = \{\alpha \in \mathfrak{t}^* \setminus \{0\}, \mathfrak{g}_\alpha \neq \{0\}\},$$

in particular, we have the following decomposition for $\mathfrak{g}$:

$$\mathfrak{g} = \mathscr{C}_\mathfrak{g}(\mathfrak{t}) \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

The elements of Φ are called the roots of $\mathfrak{g}$. We give some properties of those roots (more results can be found in [20, Chapter 8]).

Proposition 2.2.4. *1. For all $\alpha, \beta \in \mathfrak{t}^*$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$,*

2. *For all $\alpha, \beta \in \mathfrak{t}^*$ such that $\alpha + \beta \neq 0$, then the spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal with respect to the killing form $\mathscr{K}$ of $\mathfrak{g}$. In particular, the restriction of $\mathscr{K}$ to $\mathfrak{g}_0 = \mathscr{C}_\mathfrak{g}(\mathfrak{t})$ is non-degenerate.*

3. *$\mathscr{C}_\mathfrak{g}(\mathfrak{t}) = \mathfrak{t}$.*

Proof. 1. The proof is quite straightforward. Let x and y be two elements of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ respectively and $h \in \mathfrak{t}$. Then, using Jacobi's identity, we get:

$$[h, [X, Y]] = [X, [h, Y]] - [Y, [h, X]] = (\alpha + \beta)(h)[X, Y].$$

2. Similarly, let $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ such that $\alpha + \beta \neq 0$ and $h \in \mathfrak{t}$. By using the ad-invariance of the Killing form, we get:

$$\alpha(h)\mathscr{K}(X, Y) = \mathscr{K}([h, X], Y) = -\mathscr{K}(X, [h, Y]) = -\beta(h)\mathscr{K}(X, Y),$$

   and then $(\alpha + \beta)(h)\mathscr{K}(X, Y) = 0$. Because of our assumption, $\mathscr{K}(X, Y) = 0$.

   Let $x \in \mathscr{C}_\mathfrak{g}(\mathfrak{t})$ such that $\mathscr{K}(x, \mathscr{C}_\mathfrak{g}(\mathfrak{t})) = \{0\}$. Using the orthogonality of the different eigenspaces, we get that $\mathscr{K}(x, \mathfrak{g}) = \{0\}$. It implies that $x = 0$ because $\mathscr{K}$ is non-degenerate on $\mathfrak{g}$.

3. See [20, Section 8.2].

□

Because the restriction of the Killing form to the Cartan subalgebra $\mathfrak{t}$ is non-degenerate, we identify the algebra $\mathfrak{t}$ with its dual $\mathfrak{t}^*$; more precisely, for all $\alpha \in \mathfrak{t}^*$, there exists a unique $t_\alpha \in \mathfrak{t}$ such that $\alpha(t) = \mathscr{K}(t_\alpha, t)$ for all $t \in \mathfrak{t}$.

Proposition 2.2.5. *1. Φ spans $\mathfrak{t}^*$. Moreover, if $\alpha \in \Phi$, then $-\alpha \in \Phi$,*

2. *For all $\alpha \in \Phi$, $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, we have $[x, y] = \mathscr{K}(x, y)t_\alpha$,*

3. *If $\alpha \in \Phi$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is one dimensional and $\alpha(t_\alpha) \neq 0$. Moreover, the space $\mathfrak{g}_\alpha$ is one-dimensional.*

4. *If $\alpha \in \Phi$ and X_α a non-zero element of $\mathfrak{g}_\alpha$, then there exists $Y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $X_\alpha, Y_\alpha, h_\alpha = [X_\alpha, Y_\alpha]$ span a three dimensional simple subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbb{C})$.*

Proof. 1. We assume that Φ doesn't span $\mathfrak{t}^*$. Then, it follows that there exists a non-zero element $h \in \mathfrak{t}$ such that $\alpha(h) = 0$ for all $\alpha \in \Phi$. Using that $[h, X] = \alpha(h)X$ for all $X \in \mathfrak{g}_\alpha$, we get that $[h, \mathfrak{g}_\alpha] = \{0\}$. Because $\mathfrak{t}$ is abelian, it means that $[h, \mathfrak{g}] = \{0\}$ and then $\mathbb{C}h$ is an ideal of $\mathfrak{g}$ (which is solvable). In particular, $h = 0$.

We fix $\alpha \in \mathfrak{t}^*$ and X a non-zero element of $\mathfrak{g}_\alpha$. We assume that $-\alpha \in \Phi$. Then, $\mathscr{K}(X, \mathfrak{g}_\beta) = \{0\}$ for all $\beta \in \Phi \cup \{0\}$. Then, $\mathscr{K}(X, \mathfrak{g}) = \{0\}$ and we get that $X = 0$ because $\mathscr{K}$ is non-degenerate.

2. Let $\alpha \in \Phi$, $X \in \mathfrak{g}_\alpha$, $Y \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{t}$. Then:

$$\mathscr{K}(h, [X, Y]) = -\mathscr{K}([X, h], Y) = \alpha(h)\mathscr{K}(X, Y) = \mathscr{K}(t_\alpha, h)\mathscr{K}(X, Y) = \mathscr{K}(\mathscr{K}(X, Y)t_\alpha, h),$$

and in particular, $\mathscr{K}(h, [X, Y] - \mathscr{K}(X, Y)t_\alpha)$ for all $h \in \mathfrak{t}$. Because the Killing form on $\mathfrak{t}$ is non-degenerate, we get that $[X, Y] = \mathscr{K}(X, Y)t_\alpha$.

3. Obviously, the previous result proved that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathbb{C}t_\alpha$. To prove the equality, we need to prove that for all $\alpha \in \Phi$, there exists $X \in \mathfrak{g}_\alpha$ and $Y \in \mathfrak{g}_{-\alpha}$ such that $\mathscr{K}(X, Y) \neq 0$. Let $X \in \mathfrak{g}_\alpha$ be a non-zero element. If for all $Y \in \mathfrak{g}_{-\alpha}$, $\mathscr{K}(X, Y) = 0$, it follows from the previous proposition that $X \in Rad(\mathscr{K})$. Then, there exists $Y \in \mathfrak{g}_{-\alpha}$ such that $\mathscr{K}(X, Y) \neq 0$.

4. Straightforward computations.

□

Remark 2.2.6. Obviously, $h_\alpha = \frac{2t_\alpha}{\mathscr{K}(t_\alpha, t_\alpha)}$ and $h_\alpha = -h_{-\alpha}$.

For all non-zero $\alpha \in \mathfrak{t}^*$, we denote by P_α the subspace of $\mathfrak{t}^*$ given by:

$$P_\alpha = \{\beta \in \mathfrak{t}^*, \mathscr{K}^*(\alpha, \beta) = 0\},$$

where $\mathscr{K}^*$ is the corresponding non-degenerate form on $\mathfrak{t}^*$ coming from $\mathscr{K}$. The subspace P_α is an hyperplane in $\mathfrak{t}^*$. We denote by σ_α the reflexion with respect to P_α.
The Weyl group of $\mathfrak{g}$ is the subgroup of $\mathrm{GL}(\mathfrak{t}^*)$ generated by the reflections $\sigma_\alpha, \alpha \in \Phi \setminus \{0\}$. We will denote it by $\mathscr{W} = \mathscr{W}(\mathfrak{g}, \mathfrak{t})$. One can easily check that $\mathscr{W} \subseteq \mathrm{O}(\mathfrak{t}^*, \mathscr{K}^*)$, i.e. the $\sigma'_\alpha s$ preserve the form $\mathscr{K}^*$. This group is clearly finite being a subgroup of $\mathscr{S}_\Phi$.
We now recall the notion of standard cyclic modules (usually called highest weight modules).

Example 2.2.7. Let's now consider a concrete example that we will use later. We have:

$$\mathrm{SL}(n, \mathbb{C}) = \{g \in \mathrm{GL}(n, \mathbb{C}), \det(g) = 1\} \qquad \mathfrak{sl}(n, \mathbb{C}) = \{X \in \mathrm{M}(n, \mathbb{C}), \mathrm{tr}(X) = 1\}.$$

We have:

$$\mathfrak{sl}(n, \mathbb{C}) = \bigoplus_{i=1}^{n-1} \mathbb{C}(E_{i,i} - E_{i+1,i+1}) \oplus \bigoplus_{1 \le i \neq j \le n} \mathbb{C}E_{i,j}.$$

We denote by $H_i = E_{i,i} - E_{i+1,i+1}$, by $\mathfrak{t}$ the subalgebra $\mathfrak{t} = \bigoplus_{i=1}^{n-1} \mathbb{C}H_i$ and by $e_i, 1 \le i \le n$ the linear form on $\mathfrak{t}' = \{\mathrm{diag}(h_1, \dots, h_n), h_1, \dots, h_n \in \mathbb{C}\}$ given by:

$$e_i(\mathrm{diag}(h_1, \dots, h_n)) = h_i \qquad (1 \le i \le n).$$

Obviously, we get $[h_i, E_{j,k}] = (e_j - e_k)(h_i)E_{j,k}$. In particular, we get:

$$\Phi = \Phi(\mathfrak{g}, \mathfrak{t}) = \{e_i - e_j, 1 \le i \neq j \le n\}.$$

Obviously, we get that $\mathscr{W} \approx \mathscr{S}_n$.

Remark 2.2.8. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. Let $\Phi = \Phi(\mathfrak{g}, \mathfrak{t})$ be the set of roots. There exists a subset $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ of Φ such that:

- $\Delta = \Delta(\mathfrak{g}, \mathfrak{t})$ is a basis of $\mathfrak{t}^*$,
- Each root $\beta \in \Phi$ can be written as $\beta = \sum_{\alpha \in \Phi} k_\alpha \alpha$ with $k_\alpha \in \mathbb{Z}$, which are all nonnegative or all nonpositive.

The elements of Δ are called simple roots. We define by Φ^+ (resp. Φ^-) the subset of roots $\alpha \in \Phi$ such that the $k'_\alpha s$ are positive (resp. negative).

2.2.2 parameterization of finite-dimensional irreducible modules

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$. For every finite dimensional representation (V, Π) of $\mathfrak{g}$, we get, according to Theorem 2.2.3, that the elements $\Pi(h), h \in \mathfrak{t}$ form a commutating family of semisimple operators of $\mathrm{End}(V)$. In particular, we can diagonalise those operators simultaneously and we get a decomposition of the space V of the form

$$V = \bigoplus_{\lambda \in \mathfrak{t}^*} V_\lambda, \qquad V_\lambda = \{v \in V, \Pi(h)v = \lambda(h)v,\ h \in \mathfrak{t}\}. \tag{2.4}$$

We will denote by V_Π the subspace of $\mathfrak{t}^*$ given by $V_\Pi = \{\lambda \in \mathfrak{t}^*, V_\lambda \neq \{0\}\}$. The elements of V_Π are called the weights of the representation.
If the dimension of Π is infinite, the decomposition of V given in Equation (2.4) is no longer valid in general. What we can say is that the subspace W of all the weight spaces V_λ is still direct, but we have no assurance that $W = V$.
We can just notice that for all $\alpha \in \Phi$ and $\lambda \in V_\Pi$, $\Pi(\mathfrak{g}_\alpha)(V_\lambda) \subseteq V_{\alpha+\lambda}$. Indeed, for $X \in \mathfrak{g}_\alpha, v \in V_\lambda$ and $h \in \mathfrak{t}$, we get:

$$\begin{aligned} \Pi(h)(\Pi(X)v) &= \Pi(h)(\Pi(X)v) - \Pi(X)(\Pi(h)v) + \Pi(X)(\Pi(h)v) = \Pi([h,X])v + \Pi(X)(\Pi(h)v) \\ &= (\alpha(h) + \lambda(h))v. \end{aligned}$$

A non-trivial vector v is called maximal of weight λ if $v \in V_\lambda$ and if it satisfies $\Pi(X_\alpha)(v) = 0$ for all $X_\alpha \in \mathfrak{g}_\alpha, \alpha > 0$ (see Proposition 2.2.5).
Using Lie's theorem, it's clear that if the space V is finite dimensional, the existence of a maximal vector is guarantee, but it is no longer true if the space V is infinite dimensional.
A $\mathfrak{g}$-module (V, Π) is called standard cyclic if there exists a (maximal) vector $v \in V$ such that $V = \mathscr{U}(\mathfrak{g})v$. We will focus our attention on those modules. As recalled before, all the irreducible finite dimensional modules are of this form, but it also contains some infinite dimensional representations. As mentionned before, it's not guarantee for infinite dimensional representation that V is equal to the direct sum of its weight space, but this is true for cyclic modules. Indeed, we get the following theorem:

Theorem 2.2.9. *Let V be a standard cyclic $\mathfrak{g}$-module of $\mathfrak{g}$ with maximal vector $v \in V_\lambda$ with highest weight $\lambda \in \mathfrak{t}^*$.*

1. *V is spanned by the vectors $y_{\alpha_1}^{i_1} \dots y_{\alpha_n}^{i_n} v$ where i_j are all positive integers. In particular, V is the direct sum of its weight spaces and all the weights of V are of the form $\mu = \lambda - \sum_{i=1}^{n} n_i \alpha_i$, where the n_i's are all positive integers.*

2. *For each $\mu \in V_\Pi$, the dimension of the space V_μ is finite and* $\dim_{\mathbb{C}}(V_\lambda) = 1$.

3. *V in an indecomposable $\mathfrak{g}$-module, with a unique maximal (proper) submodule and a corresponding unique irreducible quotient.*

4. *Assume that V is irreducible. Then , up to a constant, the vector v is unique.*

Proof. Let's just say few words quickly about the first point (we haven't recalled in those notes the so called Poincare-Birkhoff-Witt Theorem, which says that $\mathscr{U}(\mathfrak{g})$ is isomorphic to $S(\mathfrak{g})$ as a vector space). Using the decomposition $\mathfrak{g} = \eta^- \oplus \mathfrak{b}$, we get

$$\mathscr{U}(\mathfrak{g})v = \mathscr{U}(\eta^-).\mathscr{U}(\mathfrak{b})v = \mathscr{U}(\eta^-)(\mathbb{C}v).$$

and using the PBW theorem, we get that a basis of $\mathscr{U}(\eta^-)$ given by $y_{\beta_1}^{i_1} \dots y_{\beta_m}^{i_m}$, where y_{β_k} is a non-trivial element in $\mathfrak{g}_{\beta_k}, \beta_1, \dots, \beta_m \in \Phi^-$. By writing all the β_i as a sum of positive roots with coefficients in $\mathbb{Z}^+$, we get the result.

□

Remark 2.2.10. Let's consider the algebra $\mathfrak{b}$ given by $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. We get directly that $\mathfrak{b}$ is a subalgebra of $\mathfrak{g}$, usually called Borel subalgebra of $\mathfrak{g}$.
Keeping the notations introduced before, it follows that the space $\mathbb{C}.v = V_\lambda$ is a $\mathfrak{b}$-module (obviously irreducible).

We are now faced to the natural following questions:

- Can we say that two standard cyclic modules of highest $\lambda \in \mathfrak{h}^*$ are isomorphic?
- Do there exists a cyclic module of highest weight $\lambda \in \mathfrak{h}^*$ for all $\lambda \in \mathfrak{h}^*$?

For the first one, the answer is yes if both of them are irreducible. For the second question, the answer is also positive but even more precise: we can construct those cyclic modules. The construction is due to Verma and we will recall that right now.
Let λ be an arbitrary linear form on $\mathfrak{h}$. We denote by V_λ the one-dimensional space generated formally by a vector v. Motivated by Remark 2.2.10, we define an action of $\mathfrak{b}$ on V_λ by

$$\left(h + \sum_{\alpha>0} z_\alpha\right).v := h.v = \lambda(h)v \qquad (h \in \mathfrak{t}, z_\alpha \in \mathfrak{g}_\alpha).$$

The space V_λ is clearly a $\mathfrak{b}$ (and then $\mathscr{U}(\mathfrak{b})$) module. Using that the enveloping algebra $\mathscr{U}(\mathfrak{g})$ is a $\mathscr{U}(\mathfrak{g})$-module (acting naturally on the left), we get that the following tensor $Z(\lambda)$:

$$Z(\lambda) = \mathscr{U}(\mathfrak{g}) \otimes_{\mathscr{U}(\mathfrak{b})} V_\lambda$$

is a $\mathscr{U}(\mathfrak{g})$-module.

Proposition 2.2.11. *The $\mathscr{U}(\mathfrak{g})$-module $Z(\lambda)$ is cyclic standard of highest weight λ. Moreover, there exists a unique maximal submodule $Y(\lambda)$ of $Z(\lambda)$ and the corresponding quotient $V(\lambda) = Z(\lambda)/Y(\lambda)$ is irreducible.*

Remark 2.2.12. Our motivation is to parametrize all the irreducible representations of a compact Lie group. In particular, there are all finite-dimensional, so it turns out that the next natural question is to find out which $V(\lambda)$ is finite-dimensional among all the λ's in $\mathfrak{t}^*$. The question of determining which λ corresponds to a representation of G will be treated in the next section.

Let (V, Π) be an irreducible finite-dimensional representation of $\mathfrak{g}$ (of highest weight λ). Let α be a positive root and h_α be the corresponding element of $\mathfrak{t}$ defined in Remark 2.2.6. We denote by $\{x_\alpha, y_\alpha, h_\alpha\}$ the corresponding triple of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ (we denote by $\mathfrak{s}_\alpha$ the corresponding algebra). By restricting Π to $\mathfrak{s}_\alpha$, we get that V is a $\mathfrak{s}_\alpha \approx \mathfrak{sl}(2,\mathbb{C})$-module. According to [20, Section 7], we get that $\lambda(h_\alpha)$ is a non-negative integer. Obviously, this is true for every positive roots. Now the idea is to see that this property parametrise the weights giving the irreducible finite-dimensional modules.

Definition 2.2.13. A form $\lambda \in \mathfrak{t}^*$ is said algebraically integral if $\lambda(h_\alpha) \in \mathbb{Z}$ for all $\alpha \in \Phi^+$ and is said dominant algebraically integral if $\lambda(h_\alpha) \in \mathbb{Z}^+$. We will denote by Λ^+ the lattice of dominant algebraically integral weights.

Remark 2.2.14. More generally, a weight $\lambda \in \mathfrak{t}^*$ is said dominant if $\lambda(h_\alpha) \in \mathbb{Z}^+$ for all the simple roots $\alpha \in \Delta(\mathfrak{g}, \mathfrak{t})$, and strongly dominant if the arey all strictly positive.

We get the following theorem:

Theorem 2.2.15. *If $\lambda \in \Lambda^+$, then the Verma module $V(\lambda)$ is finite dimensional and the map $\Lambda^+ \ni \lambda \to V(\lambda) \in \hat{\mathfrak{g}}_{\text{fin}}$ is one-one, where $\hat{\mathfrak{g}}_{\text{fin}}$ is the set of irreducible finite-dimensional representations of $\mathfrak{g}$. The set of weights Π_λ is permuted by $\mathscr{W}$ with $\dim(V_\mu) = \dim(V_{\sigma(\mu)})$ for all $\sigma \in \mathscr{W}$.*

2.3 Highest weight theorem and Weyl's character formula

To simplify the results we are gonna state here, we will assume throughout this section that the compact Lie group G is connected. Obviously, this assumption can be omitted and we can prove similar results for compact Lie groups with finitely many connected components.

Lemma 2.3.1. *The Lie algebra $\mathfrak{g}_0$ is reductive. In particular, $\mathfrak{g}_0 = \mathscr{Z}(\mathfrak{g}_0) \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$, where the bracket $[\mathfrak{g}_0, \mathfrak{g}_0]$ is semisimple.*

Remark 2.3.2.

1. Let $\Pi : \mathrm{G} \to \mathrm{GL}(V)$ be an irreducible representation of G. Then, the differential $d\Pi : \mathfrak{g} \to \mathrm{End}(V)$ is an irreducible representation of $\mathfrak{g}_0$, and we get the following relation:

$$\Pi \circ \exp_{\mathrm{G}} = \exp_{\mathrm{GL}(V)} \circ d\Pi : \mathfrak{g}_0 \to \mathrm{GL}(V).$$

   Then, the map $d\Pi$ can be extended to $\mathfrak{g}$ and we get a representation of the Lie algebra $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathscr{Z}(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ semisimple. In particular, the restriction of $d\Pi$ to $[\mathfrak{g}, \mathfrak{g}]$ is still irreducible and is an highest module.

2. Let $\mathfrak{g}$ be a reductive Lie algebra over $\mathbb{C}$. In particular, $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathscr{Z}(\mathfrak{g})$. Let $\mathfrak{t}$ be a Cartan subalgebra of $[\mathfrak{g}, \mathfrak{g}]$. Then, $\mathfrak{t}' = \mathfrak{t} \oplus \mathscr{Z}(\mathfrak{g})$ is a Cartan subalgebra of $\mathfrak{g}$. We have the following decomposition:

$$\mathfrak{g} = \mathfrak{t}' \oplus \bigoplus_{\alpha \in \Phi([\mathfrak{g},\mathfrak{g}],\mathfrak{t})} [\mathfrak{g}, \mathfrak{g}]_\alpha,$$

where $[\mathfrak{g},\mathfrak{g}]_\alpha = \{X \in [\mathfrak{g},\mathfrak{g}], [h,X] = \alpha(h)X\,(\forall h \in \mathfrak{t})\}$. By extending all the forms $\alpha \in \Phi([\mathfrak{g},\mathfrak{g}],\mathfrak{t})$ by 0 on $\mathscr{Z}(\mathfrak{g})$, we get that $\alpha \in \mathfrak{t}'^*$, and then,

$$\mathfrak{g} = \mathfrak{t}' \oplus \bigoplus_{\alpha\in\Phi(\mathfrak{g},\mathfrak{t}')} \mathfrak{g}_\alpha, \tag{2.5}$$

where $\mathfrak{g}_\alpha = \{X \in \mathfrak{g}, [h,X] = \alpha(h)X\,(\forall h \in \mathfrak{t}')\}$.

A torus T of G is a compect connected abelian subgroup of G (as proved in [22, Corollary 1.103], T is a product of circles)

Lemma 2.3.3. *Let* T *be a maximal torus of* G*. Then, its Lie algebra* $\mathfrak{t}_0$ *is a Cartan subalgebra of* $\mathfrak{g}_0$.

We now recall some properties of those torus and Cartan subalgebras.

Theorem 2.3.4.

1. *Two maximal abelian subalgebras of* $\mathfrak{g}_0$ *are conjugate via* $\mathrm{Ad}(\mathrm{G})$*. In particular, two maximal torus of* G *are conjugate.*
2. *If G is a compact connected Lie group and* T *is a maximal torus, then each element of* G *is conjugate to a member of* T*. In particular, every element of a compact connected Lie group* G *lies in some maximal torus.*

From now on, we fix a maximal torus T of G. As we recalled previously, every irreducible representation of G gives rise to an irreducible representation of $\mathfrak{g}$, completely determined by an algebraically integral form $\lambda \in \mathfrak{t}^*$. But what we would like to know is how to parameterize the forms coming from a representation of G among all the algebraically integral forms.

Definition 2.3.5. A linear form $\lambda \in \mathfrak{t}^*$ is called <u>analytically integral</u> if there exists a multiplicative character ξ_λ of T such that $\xi_\lambda(\exp(H)) = e^{\lambda(H)}$ for all $H \in \mathfrak{t}_0$.

Remark 2.3.6.

1. If $\lambda \in \mathfrak{t}^*$ is analytically integral, then for all $H \in \mathfrak{t}_0$ such that $exp(H) = 1$, $\lambda(H) \in 2i\pi\mathbb{Z}$. One can prove that those two conditions are equivalent.
2. Every root $\alpha \in \Phi = \Phi(\mathfrak{g},\mathfrak{t})$ is analytically integral.

One can prove that every analytically integral form is algebraically integral. We get the following theorem:

Theorem 2.3.7. *Let* G *be a compact connected group and* T *be a maximal torus in* G*. Then there is a one-one correspondence between the irreducible representations of* G *and the dominant analytically integral forms on* $\mathfrak{t}$.

The last result of this section we are going to recall is the Weyl's character formula. This formula gives a really explicit formula of the character of an highest weight representation (Π_λ, V_λ) of G with highest weight λ (analytically integral as recalled before). More precisely:

Theorem 2.3.8. *By keeping the notations of the previous paragraph, the character of the representation* Π_λ *on the torus* T *is given by the formula:*

$$\Theta_{\Pi_\lambda}(t) = \sum_{\omega \in \mathscr{W}} \frac{\varepsilon(\omega)\xi_{\omega(\lambda+\rho)-\rho}}{\prod_{\alpha \in \Phi^+} (1 - \xi_{-\alpha}(t))}. \tag{2.6}$$

Proof. For the proof of this result, one can check [22, Chapter 5]. An easy proof of this result for $\mathrm{G} = \mathrm{U}(n, \mathbb{C})$ can be found in [25, Section 6.4].

□

Chapter 3

Quasi-simple Representations of a real reductive Lie group

3.1 Notations

In this chapter, G will be a real semisimple Lie group (not necessary connected in general). We will denote by $\mathfrak{g}_0$ its Lie algebra and by its complexification.
As in Section 2.2, we will denote by $\mathscr{K}$ the Killing form on $\mathfrak{g}$.

Definition 3.1.1. We say thata. real form $\mathfrak{g}_k$ of $\mathfrak{g}$ is compact if $\mathscr{K}(X, X) < 0$ for every $X \in \mathfrak{g}_k, X \neq 0$.

We now recall a famous result of Cartan (see [22]).

Theorem 3.1.2. *There exists a compact real form $\mathfrak{g}_k$ and an automorphism θ of order 2 on $\mathfrak{g}$ with the following properties:*

1. $\theta(\mathfrak{g}_0) \subseteq \mathfrak{g}_0$ *and* $\theta(\mathfrak{g}_k) \subseteq \mathfrak{g}_k$,
2. $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ *and* $\mathfrak{g}_k = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$, *where* $\mathfrak{k}_0$ *and* $\mathfrak{p}_0$ *are the eigenspaces corresponding to the eigenvalue* 1 *and* -1 *respectively.*

We denote by $\mathfrak{k}$ and $\mathfrak{p}$ the complexifications of $\mathfrak{k}_0$ and $\mathfrak{p}_0$. From the equality $\theta([X, Y]) = [\theta(X), \theta(Y)]$ for $X, Y \in \mathfrak{g}$, we get:

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k} \qquad [\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p} \qquad [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}.$$

Exercice 1. Prove that the Lie algebra $\mathfrak{k}$ is reductive.

Let $\mathfrak{h}_{\mathfrak{p}_0}$ an abelian subalgebra of $\mathfrak{g}_0$ containing $\mathfrak{p}_0$ (of maximal dimension) and by $\mathfrak{h}_0$ an extension of $\mathfrak{h}_{\mathfrak{p}_0}$ to a maximal abelian subalgebra of $\mathfrak{g}_0$.

Lemma 3.1.3. *The subalgebra* $\mathfrak{h}_0$ *is* θ*-stable.*

Proof. Because $\mathfrak{h}_0$ is abelian, we get $[\mathfrak{h}_0, \mathfrak{h}_0] \subseteq \mathfrak{h}_0$. Moreover, for all $X \in \mathfrak{h}_0$ and $Y \in \mathfrak{h}_{\mathfrak{p}_0}$, we get $[X, Y] = 0$ and then $0 = \theta([X, Y]) = [\theta(X), \theta(Y)] = -[\theta(X), Y]$. In particular,

$$[X, Y] - [\theta(X), Y] = [X - \theta(X), Y] = 0 \qquad Y \in \mathfrak{h}_{\mathfrak{p}_0}. \tag{3.1}$$

Obviously, $X - \theta(X) \in \mathfrak{p}_0$ because $\theta(X - \theta(X)) = \theta(X) - \theta^2(X) = -(X - \theta(X))$. Then, by assumption of maximality of $\mathfrak{h}_{\mathfrak{p}_0}$ in $\mathfrak{p}_0$, $X - \theta(X) \in \mathfrak{h}_{\mathfrak{p}_0}$ (because it commutes with $\mathfrak{h}_{\mathfrak{p}_0}$ according to Equation (3.1)).
Finally, $X - \theta(X) \in \mathfrak{h}_{\mathfrak{p}_0}$, so $\theta(X) \in X + \mathfrak{h}_{\mathfrak{p}_0} \subseteq \mathfrak{h}_0$.

□

The complexification $\mathfrak{h}$ of $\mathfrak{h}_0$ is a Cartan subalgebra of $\mathfrak{g}$. As in Section 2.3, we will denote by $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$ the set of roots corresponding to $(\mathfrak{g}, \mathfrak{h})$, by $\Phi^+ = \phi(\mathfrak{g}, \mathfrak{h})$ the set of positive roots and by X_α a generator of the one-dimensional space $\mathfrak{g}_\alpha, \alpha \in \phi^+$.
Finally, we define η the subspace of $\mathfrak{g}$ spanned by the $X'_\alpha s, \alpha \in \Phi^+$ and let η_0 be the real vector space given by $\eta_0 = \eta \cap \mathfrak{g}_0$.

3.2 Garding space

Until the end of this chapter, we will denote by:

- $\mathscr{H}$ a Banach space whose norm will be denoted by $\|\cdot\|$,
- $\mathscr{B}(\mathscr{H})$ the space of bounded linear operators on $\mathscr{H}$ (we recall that A is bounded on $\mathscr{H}$ if there exists $M > 0$ such that $\|A(v)\| \leq M\|v\|$ for all $v \in \mathscr{H}$),
- N the norm on $\mathscr{B}(\mathscr{H})$ given by $N(A) = \sup_{\|\Psi\|\leq 1} \|A(\Psi)\|$, where $A \in \mathscr{B}(\mathscr{H})$,
- $(\Pi, \mathscr{H})$a continuous representation of G, i.e.

  $$\mathrm{G} \times \mathscr{H} \ni (g, \Psi) \to \Pi(g)(\Psi) \in \mathscr{H}$$

  is continuous (in particular, the map $g \to N(\Pi(g))$ is bounded on every compact set of G).

One of the main difference between finite and infinite dimensional representations comes from the fact that the Lie algebra $\mathfrak{g}_0$ of G does not act on $\mathscr{H}$ in general, i.e.

$$\lim_{t \to 0} \frac{\Pi(\exp(tX))(\Psi) - \Psi}{t}$$

does not exist for every $\Psi \in \mathscr{H}$.

We denote by $\mathscr{C}_c^\infty(\mathrm{G})$ the space of smooth and compactly supported functions on G and let dg be a left Haar measure on G.

Lemma 3.2.1. *For all $f \in \mathscr{C}_c^\infty(\mathrm{G})$, the operator $\Pi(f)$ given by*

$$\Pi(f) = \int_{\mathrm{G}} f(g)\Pi(g)dg : \mathscr{H} \to \mathscr{H}$$

is a bounded (linear) operator on $\mathscr{H}$.

Proof. Fix $f \in f \in \mathscr{C}_c^\infty(\mathrm{G})$ and $\mathscr{U} = \mathrm{supp}(f)$ the support of f. Because of the compactness of $\mathscr{U}$, we get for every $\Psi \in \mathscr{H}$ and $g \in \mathrm{G}$ that:

$$||\Pi(g)\Psi|| \leq N_g||\Psi|| \leq N_{\mathscr{U}}||\Psi|| \qquad N_{\mathscr{U}} = \sup_{g\in\mathscr{U}} N_g < \infty.$$

Then,

$$\begin{aligned} ||\Psi(f)(\Psi)|| &= ||\int_{\mathrm{G}} f(g)\Pi(g)(\Psi)dg|| \leq \int_{\mathrm{G}} |f(g)|||\Pi(g)(\Psi)||dg \\ &\leq \int_{\mathrm{G}} |f(g)|N_{\mathscr{U}}||\Psi||dg \leq (N_{\mathscr{U}}||f||_{L^1(\mathrm{G})})||\Psi|| \end{aligned}$$

Finally, because $f \in \mathrm{L}^1(\mathrm{G})$, we get that $\Psi(f)$ is bounded.

□

Definition 3.2.2. We denote by $\mathrm{Gar}(\Pi, \mathscr{H})$ the subspace of $\mathscr{H}$ spanned by the elements $\Pi(f)\Psi, f \in \mathscr{C}_c^\infty(\mathrm{G}), \Psi \in \mathscr{H}$.

Obviously, because the measure dg is left G-invariant, we get for every $f \in \mathscr{C}_c^\infty(\mathrm{G}), \Psi \in \mathscr{H}$ and $g \in \mathrm{G}$ that:

$$\begin{aligned} \Pi(g)\,(\Pi(f)(\Psi)) &= \int_{\mathrm{G}} f(h)\Pi(gh)(\Psi)dh = \int_{\mathrm{G}} f(g^{-1}h)\Pi(h)(\Psi)dh \\ &= \int_{\mathrm{G}} f \circ L_{g^{-1}}(h)\Pi(h)(\Psi)dh = \Pi(f \circ L_{g^{-1}})(\Psi) \end{aligned}$$

where $L_g : \mathrm{G} \ni h \to gh \in \mathrm{G}$ is the left translation on G. In particular, G acts on $\mathrm{Gar}(\Pi, \mathscr{H})$.

Theorem 3.2.3. *For every $X \in \mathfrak{g}_0$ and $\Psi \in \mathrm{Gar}(\Pi, \mathscr{H})$, the limit*

$$\lim_{t\to 0} \frac{\Pi(\exp(tX))(\Psi) - \Psi}{t}$$

exists and lies in $\mathrm{Gar}(\Pi, \mathscr{H})$.

For the proof of this result, one can check [2]. In particular, we get a representation of $\mathfrak{g}_0$ on $\mathrm{Gar}(\Pi, \mathscr{H})$, which can be extended to a representation of $\mathfrak{g}$ and $\mathscr{U}(\mathfrak{g})$ as usually.

Remark 3.2.4. 1. The space $\mathrm{Gar}(\Pi, \mathscr{H})$ is open and dense in $\mathscr{H}$. In particular, even if G acts on it and $\mathrm{Gar}(\Pi, \mathscr{H}) \subset \mathscr{H}$, it does not imply that Π is not irreducible.

2. Harish=Chandra pointes out that this space is not the right one to consider. Indeed, if U is a $\mathfrak{g}_0$-invariant subspace of $\mathrm{Gar}(\Pi, \mathscr{H})$, its closure $\bar{U}$ in $\mathscr{H}$ is not G-invariant in general. That's why he introduced the space of analytic vectors for such a representation $(\Pi, \mathscr{H})$ (he used the term of well-behaved vectors in [4]).

3.3 Harish-Chandra space of analytic vectors

3.3.1 Series in a Banach spaces

Definition 3.3.1. Let $\{\Psi_\alpha\}_{\alpha\in J}$ be an indexed set of elements of $\mathscr{H}$. We say that the series $\sum_{\alpha\in J} \Psi_\alpha$ converges if there exists $\phi \in \mathscr{H}$ such that for any $\varepsilon > 0$, there exists a finite. subset F_0 of J such that $\| \mathrm{S}_F - \phi \| < \varepsilon$ for every finite subset F of J containing F_0, where $\mathrm{S}_F = \sum_{\alpha\in F} \Psi_\alpha$.

We define power series with coefficients in $\mathscr{H}$ (we used the notation of [4]). We consider series of the form

$$\sum_{e_1,\dots,e_n\geq 0} \Psi(e_1, \dots, e_n) t_1^{e_1} \dots t_n^{e_n} \qquad (\Psi(e_1, \dots, e_n) \in \mathscr{H}, t_i \in \mathbb{C}, 1 \leq i \leq n). \tag{3.2}$$

Clearly, if the power series defined in Equation (3.2) is convergent for $t_1 = a_1, \dots, t_n = a_n$, $a_i \in \mathbb{C}$, it also converges (absolutely) for every t_i such that $|t_i < a_i$.

Lemma 3.3.2. *Let $\sum_{e\geq 0} \Psi(e) t^e$ be a power series (with $\Psi(e) \in \mathscr{H}$) which converges to $f(t) \in \mathscr{H}$ if $|t| < r$, with r a strictly positive number. Then, the power series $\sum_{e\geq 1} e\Psi(e) t^{e-1}$ is convergent for $|t| < r$, and its sum is equal to the limit*

$$\lim_{h\to 0} \frac{f(t+h) - f(t)}{h} \qquad |t| < r.$$

We can generalize this for power series depending on more than one variable.

Corollary 3.3.3. *Let $\sum_{e_1,\dots,e_n\geq 0} \Psi(e_1, \dots, e_n) t_1^{e_1} \dots t_n^{e_n}$ (as in Equation (3.2)) be a power series which converges to $f(t) = f(t_1, \dots, t_n)$ if $|t_i| < r_i$ ($r_i > 0, 1 \leq i \leq n$).*
Then, the series $\sum_{e_1,\dots,e_n\geq 0} \Psi(e_1, \dots, e_n) \frac{\partial}{\partial t_i} \prod_{j=1}^{n} t_j^{e_j}$ converges in the same region to

$$\frac{\partial}{\partial t_i} f(t_1, \dots, t_n) = \lim_{h\to 0} \frac{f(t_1, \dots, t_i + h, \dots, t_n) - f(t_1, \dots, t_n)}{h}.$$

Exercice 2. Prove the Lemma 3.3.2 and Corollary 3.3.3.

3.3.2 Analytic maps on manifolds

Let M be a real manifold and $f : M \to \mathscr{H}$ a map.

Definition 3.3.4. The map f is called analytic at a point $x_0 \in M$ if for every coordinates system $(t_1, \ldots, t_n)$ at x_0 such that $t_i(x_0) = 0$ for every $i \in [|1, n|]$, there exists a neighbourhood V of x_0 in M (on which out coordinates are valid) and a power series $\sum_{e_1,\ldots,e_n \geq 0} \Psi(e_1, \ldots, e_n) t_1^{e_1} \ldots t_n^{e_n}$ (as in Equation (3.2)) such that the series converges to $f(x)$ for $t_i = t_i(x)$, $x \in V$.

Remark 3.3.5. 1. If the condition of Definition 4.5.1 is satisfied for one coordinate system $(t_1, \ldots, t_n)$, then it holds for all coordinate systems $(u_1, \ldots, u_n)$ such that $u_i(x_0) = 0, 1 \leq i \leq n$.

2. A map $f : M \to \mathscr{H}$ is called analytic if it is analytic at every point of M.

3. Let $f : M \to \mathscr{H}$ analytic at $x_0 \in M$ and $u : M \to \mathbb{C}$ also analytic at x_o. Then, uf is analytic at x_0. Indeed, because f is analytic, there exists a system coordinate $(t_1, \ldots, t_n)$ on an open subset $\mathscr{U}_1$ of M such that

$$f(x) = \sum_{e_1,\ldots,e_n \geq 0} \Psi(e_1, \ldots, e_n) t_1^{e_1}(x) \ldots t_n^{e_n}(x) \qquad (x \in \mathscr{U}_1).$$

Using the first remark, there exists an open neighbourhood $\mathscr{U}_2$ of x_0 in M such that

$$u(x) = \sum_{f_1,\ldots,f_n \geq 0} a(f_1, \ldots, f_n) t_1^{f_1}(x) \ldots t_n^{f_n}(x) \qquad (x \in \mathscr{U}_2, a(f_1, \ldots, f_n) \in \mathbb{C}).$$

In particular, for every $x \in \mathscr{U}_3 = \mathscr{U}_1 \cap \mathscr{U}_2$, both power series are absolutely convergent and we get

$$uf(x) = \sum_{\substack{e_1,\ldots e_n \geq 0 \\ f_1,\ldots,f_n \geq 0}} \Psi(e_1, \ldots, e_n) a(f_1, \ldots, f_n) \prod_{i=1} t_i(x)^{e_i + f_i}.$$

4. More generally, let M be a manifold, $\mathscr{H}$ and $\mathscr{H}'$ be two complex Banach spaces and $x_0 \in M$. Fix $f : M \to \mathscr{H}$ an analytic map at x_0 and A a continuous mapping on $\mathscr{H}$ into $\mathscr{H}'$. Then, the composition $A \circ f$ is analytic at x_0.

3.3.3 Analytic maps on Lie groups

Lie groups by a particular cases of smooth manifolds, but "the translations" simplify the previous definitions.

Definition 3.3.6. For an irreducible continuous representation $(\Pi, \mathscr{H})$ of G, a vector $\Psi \in \mathscr{H}$ is said to be analytic if the map:

$$\gamma_\Psi : \mathrm{G} \ni g \to \Pi(g)\Psi \in \mathscr{H}$$

is analytic.

We get the following lemma.

Lemma 3.3.7. *Let $\Psi \in \mathscr{H}$ such that the map γ_Ψ is analytic at $g = e$. Then, Ψ is analytic.*

Proof. Let $g \in \mathrm{G}$. The left translation L_g is analytic. Moreover, for every $h \in \mathrm{G}$, we get:

$$\Pi(g) \circ \gamma_\Psi \circ L_g(h) = \Pi(g)(\gamma_\Psi(g^{-1}h)) = \Pi(g)(\Pi(g^{-1}h)\Psi) = \Pi(h)\Psi = \Gamma_\psi(h).$$

By taking $h = g$ and using that γ_Ψ is analytic at $g = e$, we get that γ_Ψ is analytic everywhere.

□

Notation 3.3.8. We denote by $\mathrm{Har}(\Pi, \mathscr{H})$ the space of analytic vectors of the representation $(\Pi, \mathscr{H})$ of G.

Exercice 3. Prove that for all $\Psi \in \mathrm{Har}(\Pi, \mathscr{H})$ and $X \in \mathfrak{g}_0$, the limit

$$\lim_{t\to 0} \frac{\Pi(\exp(tX))\Psi - \Psi}{t}$$

exists (we denote by $d\Pi(X)$ this limit). Moreover, $d\Pi(X)\Psi$ is in $\mathrm{Har}(\Pi, \mathscr{H})$ and the map $d\Pi$ satisfies

$$d\Pi([X, Y]) = d\Pi(X)d\Pi(Y) - d\Pi(Y)d\Pi(X)$$

for all $X, Y \in \mathfrak{g}_0$. In particular, $(d\Pi, \mathrm{Har}(\Pi, \mathscr{H}))$ is a representation of $\mathfrak{g}_0$.

We will now study some properties of the Harish-Chandra space. In particular, we will first prove that it solves the problem we mentionned before with the Garding space (see Remark 3.2.4). The fact that $\mathrm{Har}(\Pi, \mathscr{H})$ is dense in $\mathscr{H}$ will be studied later.

Theorem 3.3.9. *Let Ψ be an element of $\mathrm{Har}(\Pi, \mathscr{H})$. Then, there exists a neighbourhood V of 0 in $\mathfrak{g}_0$ such that the series $\sum_{m\geq 0} \frac{1}{m!} d\Pi(X^m)\Psi$ converges to $\Pi(\exp(X))\Psi$ for $X \in V$.*

Proof. □

We get the following corollary.

Corollary 3.3.10. *Let $\Psi \in \mathrm{Har}(\Pi, \mathscr{H})$ and let $\overline{d\Pi(\mathscr{U}(\mathfrak{g})\Psi}$ the closure of $d\Pi(\mathscr{U}(\mathfrak{g})\Psi$ in $\mathscr{H}$. Then, the space $\overline{d\Pi(\mathscr{U}(\mathfrak{g})\Psi}$ is G-invariant.*

Recall 3.3.11. We recall here a consequence of the so-called Hahn-Banach's Theorem. Let V be a normed vector space and U a linear subspace of V (not necessarily closed). If z is an element of V not in the closure of U, then there exists a continuous linear map $\Psi : V \to |mathbbR$ such that $\Psi(X) = 0$ for every $X \in U$, $\Psi(z) = 1$ and $||\Psi|| = dist(z, U)^{-1}$.

Proof of Corollary 3.3.10. Let Ψ_0 be an element of $d\Pi(\mathscr{U}(\mathfrak{g})\Psi$. We want to prove that $\Pi(g)\Psi_0 \in \overline{d\Pi(\mathscr{U}(\mathfrak{g})\Psi}$ for all $g \in \mathrm{G}$. According to Recall 3.3.11, it is enough to prove for all linear form ϕ on $\mathscr{H}$ such that $\phi_{|d\Pi(\mathscr{U}(\mathfrak{g})\Psi} = 0$, we have $\phi(\Pi(g)\Psi_0) = 0$.
Fix such a function ϕ. For all $X \in$, the function

$$\mathrm{G} \ni g \to \Pi(g)d\Pi(X)\Psi \in \mathscr{H} \qquad (\Psi \in \mathrm{Har}(\Pi, \mathscr{H}))$$

is analytic. Using Theorem 3.3.9, there exists a neighbourhood V of 0 in $\mathfrak{g}_0$ such that

$$\Pi(\exp(X))\Psi_0 = \sum_{m\geq 0} \frac{1}{m!} d\Pi(X^m)\Psi_0 =$$

for all $X \in V$. In particular, by linearity of ϕ, we get:

$$\phi(\Pi(\exp(X))\Psi_0) = \phi(\sum_{m\geq 0} \frac{1}{m!} d\Pi(X^m)\Psi_0) = \sum_{m\geq 0} \frac{1}{m!} \phi(d\Pi(X^m)\Psi_0).$$

By definition of Φ_0, $d\Pi(X^m)\Psi_0 \in d\Pi(\mathscr{U}(\mathfrak{g})\Psi$, so by assumptions on ϕ, we get $\phi(d\Pi(X^m)\Psi_0) = 0$.
In particular, $\phi(\Pi(\exp(X))\Psi_0) = 0$ for every $X \in V$, and it implies that $\Phi(\Pi(g)\Psi_0)$ is zero on an open neighbourhood of $e \in \mathrm{G}$. Using that the group G is connected and that the map $\mathrm{G} \ni g \to \phi(\Pi(g)\Psi_0) \in \mathbb{C}$ is analytic, we get that

$$\phi(\Pi(g)\Psi_0) = 0 \qquad (g \in \mathrm{G}). \tag{3.3}$$

The Equation (3.3) holds for every linear form ϕ such that $\phi_{|d\Pi(\mathscr{U}(\mathfrak{g})\Psi} = 0$. By fixing $g \in \mathrm{G}$ and by taking all the $\phi' s$ with such properties, we get that $\Pi(g)\Psi_0 \in \overline{d\Pi(\mathscr{U}(\mathfrak{g}))\Psi}$. In particular, we get. that

$$\Pi(g)\,(d\Pi(\mathscr{U}(\mathfrak{g})\Psi)) \subseteq \overline{d\Pi(\mathscr{U}(\mathfrak{g})\Psi)},$$

and by continuity of the $\Pi(g)'s$, it follows that $\Pi(g)\left(\overline{d\Pi(\mathscr{U}(\mathfrak{g})\Psi)}\right) \subseteq \overline{d\Pi(\mathscr{U}(\mathfrak{g})\Psi)}$. This conclude the proof.

□

We now study the question of density of $\mathrm{Har}(\Pi, \mathscr{H})$ in $\mathscr{H}$,

3.3.4 Analytic functions and Dirac sequences

Let $(\Pi, \mathscr{H})$ be a representation of G on a Banach space $\mathscr{H}$. Obviously, the function:

$$\mathrm{G} \ni g \to N(\Pi(g)) = \sum_{\|\Psi\| \leq 1} \|\Pi(g)\Psi\| \in \mathbb{C}$$

is measurable. Let dg be a Haar measure on G and $L(\mathrm{G}, \Pi)$ the space of complex valued measurable functions f such that $\|f\|_\Pi < \infty$, where

$$\|f\|_\Pi = \int_{\mathrm{G}} |f(g)| N(\Pi(g)) dg.$$

Exercice 4. Prove that $(L(\mathrm{G}, \Pi), \|\cdot\|_\Pi)$ is a Banach space (i.e. $\|\cdot\|_\Pi$ is a norm on $L(\mathrm{G}, \Pi)$ and every Cauchy sequences in $L(\mathrm{G}, \Pi)$ is convergent in $L(\mathrm{G}, \Pi)$).

We define a representation λ of G on the space $L(\mathrm{G}, \Pi)$ by:

$$\lambda(g)f(y) = f(g^{-1}y) \qquad (g, y \in \mathrm{G}, f \in L(\mathrm{G}, \Pi)).$$

Obviously, the operators $\lambda(g)$ is a bounded operator on $L(\mathrm{G}, \Pi)$. Indeed, for $g \in \mathrm{G}$ and $f \in L(\mathrm{G}, \Pi)$, we get:

$$\begin{aligned}
\|\lambda(g)f\|_\Pi &= \int_{\mathrm{G}} |\lambda(g)f(h)| N(\Pi(h)) dh = \int_{\mathrm{G}} |f(g^{-1}h)| N(\Pi(h)) dh \\
&= \int_{\mathrm{G}} |f(h)| N(\Pi(gh)) dh \leq \int_{\mathrm{G}} |f(h)| N(\Pi(h)) N(\Pi(g)) dh \\
&\leq N(\Pi(g)) \int_{\mathrm{G}} |f(h)| N(\Pi(h)) dh \leq N(\Pi(g)) \|f\|_\Pi
\end{aligned}$$

Exercice 5. Prove that the map λ is continuous.

We denote by $\mathrm{Har}(\lambda, L(\mathrm{G}, \Pi))$ the set of analytic vectors of $(\lambda, L(\mathrm{G}, \Pi))$.

Lemma 3.3.12. *For all $f \in \mathrm{Har}(\lambda, L(\mathrm{G}, \Pi))$ and $\Psi \in \mathscr{H}$. Then, the element $T_f(\Psi)$ given by*

$$T_f(\Psi) = \int_{\mathrm{G}} f(g) \Pi(g) \Psi dg$$

is in $\mathrm{Har}(\Pi, \mathscr{H})$.

Proof. First of all, because $f \in L(\Pi, \mathscr{H})$, the operator $T_f = \int_{\mathrm{G}} f(g)\Pi(g) dg$ is well-defined and the map

$$L(\mathrm{G}, \Pi) \ni f \to T_f \in \mathscr{B}(\mathscr{H})$$

is continuous. We want to prove that $\mathrm{G} \ni g \to \Pi(g)T_f(\Psi) \in \mathscr{H}$ is analytic. We have:

$$\begin{aligned}\Pi(g)T_f\Psi &= \Pi(g)\left(\int_{\mathrm{G}} f(h)\Pi(h)dh\right) = \int_{\mathrm{G}} f(h)\Pi(gh)dh \\ &= \int_{\mathrm{G}} f(g^{-1}h)\Pi(h)dh = T_{\lambda(g)f}(\Psi)\end{aligned}$$

By assumption, the map $\mathrm{G} \ni g \to \lambda(g)f \in L(\Pi, \mathscr{H})$ is analytic. Moreover, the maps

$$\lambda(g)f \to T_{\lambda(g)f}, \qquad T_{\lambda(g)f} \to T_{\lambda(g)f}\Psi$$

are continuous. It follows that the map $\mathrm{G} \ni g \to \Pi(g)T_f(\Psi) \in \mathscr{H}$ is analytic and then, $T_f(\Psi) \in \mathrm{Har}(\Pi, \mathscr{H})$.

□

Definition 3.3.13. Let μ be any measurable function on G which is real and non-negative and let $\{f_n(x)\}_{n\in\mathbb{Z}_*^+}$ be a sequence of functions in $L^1(\mathrm{G}, dg)$. The set $\{f_n(x)\}_n$ is called a Dirac μ-sequence if

1. $\int_{\mathrm{G}} f_n(g)dg = 1$ and $\lim_{n\to+\infty} \int_{\mathrm{G}} |f_n(g)|dg = 1$,
2. For any measurable neighbourhood V of 1 in G,

$$\lim_{n\to+\infty} \int_{\mathrm{G}\setminus V} |f_n(g)|(1 + \mu(g))dg = 0$$

The interest of considering such sequences is given by the following lemma.

Lemma 3.3.14. *Suppose there exists a Dirac $N(\Pi(g))$-sequence $\{f_n(g)\}_{n\in\mathbb{Z}_*^+}$ such that $f_n \in \mathrm{Har}(\lambda, L(\mathrm{G}, \mathscr{H}))$. Then, the space $\mathrm{Har}(\Pi, \mathscr{H})$ is dense in $\mathscr{H}$.*

Proof. We fix $\Psi \in \mathscr{H}$. For $\varepsilon > 0$, we can find an open neighbourhood $\mathscr{U}$ of 1 in G such that $||\Pi(g)\Psi - \Psi|| < \varepsilon||\Psi||$ (because the representation Π is continuous, so $\lim_{g\to 1} ||\Pi(g)\Psi - \Psi|| = 0$). We define $(\Psi_n)_n$ by $\Psi_n = T_{f_n}\Psi$. Because of Lemma 3.3.14, $\Psi_n \in \mathrm{Har}(\Pi, \mathscr{H})$. We now show that $\lim_{n\to\infty} \Psi_n = \Psi$. We have:

$$\begin{aligned}\Psi_n - \Psi &= \int_{\mathrm{G}} f_n(g)\Pi(g)\Psi dg - \Psi = \int_{\mathrm{G}} f_n(g)\Pi(g)\Psi dg - \left(\int_{\mathrm{G}} f_n(g)dg\right)\Psi = \int_{\mathrm{G}} f_n(g)(\Pi(g)\Psi - \Psi)dg \\ &= \int_{\mathscr{U}} f_n(g)(\Pi(g)\Psi - \Psi)dg + \int_{\mathrm{G}\setminus\mathscr{U}} f_n(g)(\Pi(g)\Psi - \Psi)dg\end{aligned}$$

Then,

$$\begin{aligned}\Psi_n - \Psi &\leq \int_{\mathscr{U}} |f_n(g)| ||\Pi(g)\Psi - \Psi|| dg + \int_{G\setminus\mathscr{U}} |f_n(g)| ||\Pi(g)\Psi - \Psi|| dg \\ &\leq \varepsilon ||\Psi|| \int_{\mathscr{U}} |f_n(g)| dg + \int_{G\setminus\mathscr{U}} |f_n(g)| ||(\Pi(g) - \mathrm{Id})\Psi|| dg \\ &\leq \varepsilon ||\Psi|| \int_{G} |f_n(g)| dg + \int_{G\setminus\mathscr{U}} |f_n(g)| (1 + N(\Pi(g))) dg\end{aligned}$$

By assumption, there exists $N_1 \in \mathbb{Z}^+$ such that for every $n \geq N_1$, we get:

$$\int_{G\setminus\mathscr{U}} |f_n(g)| (1 + N(\Pi(g))) dg < \varepsilon.$$

Similarly, there exists $N_2 \in \mathbb{Z}^+$ such that for every $n \geq N_2$,

$$\int_{G} |f_n(g)| \leq 1 + \varepsilon.$$

Then, for every $n \geq \max(N_1, N_2)$, we get:

$$||\Psi_n - \Psi|| \leq \varepsilon ||\Psi|| (1 + \varepsilon) + \varepsilon ||\Psi|| \leq 3\varepsilon ||\Psi||$$

for $\varepsilon < 1$. Then, $\mathrm{Har}(\Pi, \mathscr{H})$ is dense in $\mathscr{H}$.

□

For the study of the density of the Harish-Chandra space, we will proceed as follow:

1. We will first prove that if S is a simply connected quasi-nilpotent group, we can find such a sequence and then, the space of S-analytic vectors is dense (we recall that a Lie algebra $\mathfrak{s}$ is quasi-nilpotent if it can be written as the direct sum of an abelian subalgebra and a nilpotent ideal η, and a connected Lie group is quasi-simple if its Lie algebra is quasi-nilpotent).

2. The second case if to study the case $G = K \times S$, with K compact and S quasi-nilpotent (with another assumption concerning the dimension of the K-orbits of analytic vectors) and prove that in this context, the space of analytic vectors is dense.

3. The last step is to consider a representation of a semisimple Lie group. By Iwasawa decomposition, we get $G = K \times S$ but K is not compact in general). We prove the density of the Harish-Chandra space for a certain class of representation called permissible (see Definition 3.4.2).

We admit the following result.

Proposition 3.3.15. *Let* $(\Pi, \mathscr{H})$ *ba a continuous representation of a simply connected quasi-nilpotent group* S*. Then, the space* $\mathrm{Har}(\Pi, \mathscr{H})$ *is dense in* $\mathscr{H}$*.*

We now prove the following result.

Theorem 3.3.16. *Let* G *be a connected Lie group having to analytic subgroups* K *and* S *such that:*

1. K *is compact and* S *is quasi-nilpotent,*
2. *Every element* g *can be written uniquely as* $g = ks, k \in \mathrm{K}, s \in \mathrm{S}$*.*

Let $(\Pi, \mathscr{H})$ *be a representation of* G *and* $\mathrm{Har}(\Pi, \mathscr{H})_0$ *be the subspace of* $(\Pi, \mathscr{H})$ *consisting of analytic vectors* Ψ *such that* $\langle \Pi(k)\Psi, k \in \mathrm{K}\rangle$ *is finite-dimensional. Then,* $\mathrm{Har}(\Pi, \mathscr{H})_0$ *is dense in* $\mathscr{H}$*.*

Proof. By restriction, Π is a representation of S. In particular, the space $\mathrm{Har}(\Pi_{|\mathrm{S}}, \mathscr{H})$ is dense in $\mathscr{H}$.
Let $\Psi_0 \in \mathscr{H}$ and $\varepsilon > 0$. There exists $\Psi \in \mathrm{Har}(\Pi_{|\mathrm{S}}, \mathscr{H})$ such that $||\Psi - \Psi_0|| < \varepsilon$. Because $\lim_{g\to 1} ||\Pi(g)\Psi - \Psi|| = 0$, we can find an open neighbourhood $\mathscr{U}_\varepsilon$ of 1 in K such that $||\Pi(u)\Psi - \Psi|| < \varepsilon$ for every $u \in \mathscr{U}_\varepsilon$
We now consider a continuous function f which is real-valued, non-negative on K such that $f = 0$ outside $\mathscr{U}_\varepsilon$ and such that $\int_{\mathrm{K}} f(k)dk = 1$, where dk is normalized on K. We denote by Ψ_1 the element of $\mathscr{H}$ given by

$$\Psi_1 = \int_{\mathrm{K}} f(u)\Pi(u)\Psi du.$$

Because of our assumptions on f, Ψ_1 is well-defined and we get:

$$\Psi_1 - \Psi = \int_{\mathrm{K}} f(k)\Pi(k)\Psi dk - \left(\int_{\mathrm{K}} f(k)dk\right)\Psi = \int_{\mathscr{U}_\varepsilon} f(k)(\Pi(k) - \mathrm{Id})\Psi dk,$$

because $\mathrm{supp}(f) \subseteq \mathscr{U}_\varepsilon$. Then,

$$||\Psi_1 - \Psi|| \leq \int_{\mathscr{U}_\varepsilon} f(k)||(\Pi(k) - \mathrm{Id})\Psi||dk \leq \varepsilon \int_{\mathscr{U}_\varepsilon} f(k)dk \leq \varepsilon.$$

We now have to prove that:

1. The element $\Psi \in \mathrm{Har}(\Pi, \mathscr{H})$,
2. $\langle \Pi(k)\Psi, k \in \mathrm{K}\rangle$ is finite dimensional.

To prove the second one, we apply Peter-Weyl's Theorem ([22, Chapter 4, Section 3]). Let $\mathscr{R}$ be the set of all finite linear combinations of the coefficients of finite dimensional representations of K. Then, for every $\varepsilon_1 > 0$, there exists an element $\omega_{\varepsilon_1} \in \mathscr{R}$ such that for every $k \in \mathrm{K}$,

$$|f(k) - \omega_{\varepsilon_1}| < \varepsilon_1.$$

Then,

$$\Psi - \Psi_2 = \Psi - \Psi_1 + \Psi_1 - \Psi_2 = \int_{\mathrm{K}} f(k)(\Pi(k) - \mathrm{Id})\Psi dk + \int_{\mathrm{K}} (f(k) - \omega_{\varepsilon_1}(k))\Pi(k)\Psi dk,$$

and in particular

$$\begin{aligned}
\|\Psi - \Psi_2\| &\leq \int_{\mathrm{K}} f(k)\|(\Pi(k) - \mathrm{Id})\Psi\| dk + \int_{\mathrm{K}} |f(k) - \omega_{\varepsilon_1}(k)| \|\Pi(k)\Psi\| dk \\
&\leq \varepsilon + \varepsilon_1 \int_{\mathrm{K}} \|\Pi(k)\Psi\| dk \leq \varepsilon + \varepsilon_1 \int_{\mathrm{K}} \left(\sup_{u \in \mathrm{K}} N(\Pi(u))\right) \|\Psi\| dk \\
&\leq \varepsilon + \varepsilon_1(\|\Psi_0\| + \varepsilon) \left(\sup_{u \in \mathrm{K}} N(\Pi(u))\right) \\
&\leq 2\varepsilon
\end{aligned}$$

for ε_1 small enough. In particular, because $\omega_{\varepsilon_1} \in \mathscr{R}$, the space $\langle \Pi(k)\Psi_2, k \in \mathrm{K}\rangle$ is finite dimensional. Which proves that $\langle \Pi(k)\Psi, k \in \mathrm{K}\rangle$ is finite dimensional.

□

Exercice 6. Prove that the element Ψ constructed in the proof of Theorem 3.3.16 is in $\mathrm{Har}(\Pi, \mathscr{H})$.

Now, we will apply this result for partular representations of real semisimple connected Lie groups.

3.4 Permissible representations

Let $(\Pi, \mathscr{H})$ be a representation of G. Obviously, Π can be extended to a representation of its simply connected cover $\widetilde{\mathrm{G}}$. In particular, we can assume, without loss of generality, that G is simply connected.

We use the notations introduced in Section 3.1 and we denote by K, A and N the analytic subgroups corresponding to $\mathfrak{k}_0$, $\mathfrak{h}_{\mathfrak{p}_0}$ and η_0. To keep the previous notations, we will denote by S the subgroups $\mathrm{A\,N}$.

As proved by Iwasawa, we get $\mathrm{G} = \mathrm{K\,S}$, where S is quasi-nilpotent, but K is not compact in general (indeed, $Z(\mathrm{G}) \subseteq \mathrm{K}$ with $Z(\mathrm{G})$ discrete but not necessarily finite). We recall the following lemma.

Lemma 3.4.1. *Let $\mathfrak{c}_0 = Z(\mathfrak{k}_0)$ and D be the corresponding analytic subgroups of G. Then, $\mathrm{K}^* = \mathrm{K} / \mathrm{D} \cap Z(\mathrm{G})$ is compact. Moreover, $\mathrm{K} \approx [\mathrm{K}, \mathrm{K}] \rtimes \mathrm{D}$, where $[\mathrm{K}, \mathrm{K}]$ is the commutator subgroup of K.*

Because K is not compact, we cannot assure directly that every irreducible modules of K are finite dimensional. If we assume that the restriction of Π to $\mathrm{D} \cap Z(\mathrm{G})$ acts as a character, then, every irreducible representation (λ, V_λ) of K such that $\mathrm{Hom}_{\mathrm{K}}(V_\lambda, \mathscr{H})$ induce a representation λ^* on V_λ which is still irreducible, and then, finite dimensional using Section 2.3. In particular, this assumption on Π will allow us to use the Theorem 3.3.16. That is the motivation of the following concept.

Definition 3.4.2. The representation $(\Pi, \mathscr{H})$ is called permissible if $\Pi(z)$ is a scalar multiple of the unit operator for all $z \in \mathrm{D} \cap \mathrm{Z(G)}$.

Until the end of this section, $(\Pi, \mathscr{H})$ will be a permissible representation (not necessarily irreducible).

Notation 3.4.3. As in Theorem 3.3.16, we will denote by $\mathrm{Har}(\Pi, \mathscr{H})_0$ the subset of $\mathrm{Har}(\Pi, \mathscr{H})$ consisting of K-finite vectors.

Proposition 3.4.4. *The space* $\mathrm{Har}(\Pi, \mathscr{H})_0$ *is dense in* $\mathscr{H}$.

Proof. Clearly, there exists a linear form μ on $\mathfrak{c}_0$ such that

$$\Pi(\exp(X)) = e^{\mu(X)}\Pi(1) \qquad (X \in \mathfrak{c}_0, \exp(X) \in \mathrm{D} \cap \mathrm{Z(G)}).$$

We denote by Π^* the map defined by

$$\Pi^*(k\exp(\Gamma)) = e^{-\mu(\Gamma)}\Pi(k\exp(\Gamma))$$

where $k\exp(\Gamma), k \in [\mathrm{K}, \mathrm{K}], \Gamma \in \mathfrak{c}_0$ (using the description of K given in Lemma 3.4.1).
For every $k \in \mathrm{K}$, $\Gamma \in \mathfrak{c}_0$ and $\Gamma' \in \mathfrak{c}_0$ such that $\exp(\Gamma') \in \mathrm{D} \cap \mathrm{Z(G)}$, we get:

$$\begin{aligned}
\Pi^*(k\exp(\Gamma)\exp(\Gamma')) &= \Pi^*(k\exp(\Gamma + \Gamma')) = e^{-\mu(\Gamma+\Gamma')}\Pi(k\exp(\Gamma + \Gamma')) \\
&= e^{-\mu(\Gamma+\Gamma')}\Pi(k\exp(\Gamma))\Pi(\Gamma') = e^{-\mu(\Gamma+\Gamma')}\Pi(k\exp(\Gamma))e^{\mu(\Gamma')}\Pi(1) \\
&= e^{-\mu(\Gamma)}\Pi(k\exp(\Gamma)) = \Pi^*(k\exp(\Gamma))
\end{aligned}$$

In particular, Π^* is a well-defined map on K^*. For every $k \in \mathrm{K}$, we denote by k^* the corresponding element in K^*. For every $k, k' \in \mathrm{K}$ and $\Gamma, \Gamma' \in \mathfrak{c}_0$, we get:

$$\begin{aligned}
\Pi^*((k\exp(\Gamma))^*(k'\exp(\Gamma'))^*) &= \Pi^*((kk'\exp(\Gamma + \Gamma'))^*) = e^{-\mu(\Gamma+\Gamma')}\Pi(kk'\exp(\Gamma + \Gamma')) \\
&= e^{-\mu(\Gamma)}\Pi(k\exp(\Gamma))e^{-\mu(\Gamma')}\Pi(k\exp(\Gamma')) \\
&= \Pi^*((k\exp(\Gamma))^*)\Pi^*((k'\exp(\Gamma'))^*)
\end{aligned}$$

In particular, Π^* is a representation of K^* on the space $\mathscr{H}$.
We now fix $\Psi_0 \in \mathscr{H}$ and $\varepsilon > 0$. According to Proposition 3.3.15, there exists $\Psi \in \mathrm{Har}(\Pi_{|_\mathrm{S}}, \mathscr{H})$ such that $||\Psi - \Psi_0|| < \varepsilon$. Using Peter-Weyl's Theorem and Proposition 2.1.6, there exists a finite

linear combination ω of the coefficients of some finite dimension representation of K^* such that

$$\| \int_{\mathrm{K}^*} \omega(u^*)\Pi^*(u^*)\Psi du^* - \Psi\| < \varepsilon.$$

In particular,

$$\| \int_{\mathrm{K}^*} \omega(u^*)\Pi^*(u^*)\Psi du^* - \Psi\| \leq \| \int_{\mathrm{K}^*} \omega(u^*)\Pi^*(u^*)\Psi du^* - \Psi_0\| + \|\Psi_0 - \Psi\| < 2\varepsilon.$$

The end of the proof is let as an exercice.

□

Exercice 7. Prove that the element $\int_{|K^*} \omega(u^*)\Pi^*(u^*)\Psi du^*$ defined in the proof of Proposition lies in $\mathrm{Har}(\Pi, \mathscr{H})_0$.

Notation 3.4.5. We denote by $\Omega(\mathrm{K})$ the set of equivalence classes of finite dimensional irreducible representations $(\mathrm{D}, V_{\mathrm{D}})$ of K.

Remark 3.4.6. According to Schur's Lemma (see [22, Chapter IV, Proposition 4.8]), every representation $(\mathrm{D}, V_{\mathrm{D}})$ of K in $\Omega(\mathrm{K})$ induces a representation $(\mathrm{D}^*, V_{\mathrm{D}})$ of K^*, which is still irreducible.

For every $(\mathrm{D}, V_{\mathrm{D}}) \in \Omega(\mathrm{K})$, we denote by $\mathscr{H}_{\mathrm{D}}$ the set of all elements $\Psi \in \mathscr{H}$ such that there exists a finite dimensional linear space U containing Ψ which is invariant and semisimple under $\Pi(\mathrm{K})$ and is such that the representation of K induced on every simple subspace of U lies in D.

Remark 3.4.7. The space $\mathscr{H}_{\mathrm{D}}$ is the set of K-finite vectors in the D-isotypic component of Π.

Lemma 3.4.8. *Every element in a* D*-isotypic component is* K^**-finite (and then* K*-finite).*

Proof. Fix $\mathrm{D} \in \Omega(\mathrm{K})$ and $\mathscr{H}(\mathrm{D})$ be the D-isotypic component corresponding to $(\mathrm{D}, V_{\mathrm{D}})$. As before, we denote by D^* the corresponding representation of K^*, and we get obviously that

$$\mathscr{H}(\mathrm{D}) = \mathscr{H}(\mathrm{D}^*). \tag{3.4}$$

We denote by χ_{D} and χ_{D^*} the corresponding characters and by E_{D} the projection of $\mathscr{H}$ onto $\mathscr{H}_D$. Using [27, Lemma 1.4.6] and Equation (3.4), we get:

$$\mathrm{E}_{\mathrm{D}} = \int_{\mathrm{K}^*} \chi_{D^*}(u^*)\Pi^*(u^*)du^*, \tag{3.5}$$

where $du^*(\mathrm{K}^*) = 1$. Clearly, $\mathscr{H}(\mathrm{D}) = \mathrm{E}_{\mathrm{D}}(\mathscr{H})$. Moreover, for every $\Psi \in \mathscr{H}$ and $k^* \in \mathrm{K}^*$, we get:

$$\begin{aligned}\Pi^*(k^*)(\mathrm{E_D}(\Psi)) &= \int_{\mathrm{K}^*} \chi_{\mathrm{D}^*}(u^*)\Pi^*(k^*u^*)\Psi du^* = \int_{\mathrm{K}^*} \chi_{\mathrm{D}^*}(k^{*-1}u^*)\Pi^*(u^*)\Psi du^* \\ &= \int_{\mathrm{K}^*} L(k^*)\chi_{\mathrm{D}^*}(u^*)\Pi(u^*)\Psi du^*\end{aligned}$$

The result follows by noticing that the left translation $L(k^*)\chi_{\mathrm{D}^*}$ of the character χ_{D^*} span a finite dimensional subspace of $\mathrm{L}^2(\mathrm{K}^*, dk^*)$.

□

According to the Proposition 3.5.2, we get that the space

$$\mathrm{Har}(\Pi, \mathscr{H}) \cap \left(\sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathscr{H}_{\mathrm{D}} \right)$$

is dense in $\mathscr{H}$.

Lemma 3.4.9. *1. For every* $\mathrm{D} \in \Omega(\mathrm{K})$*, we denote by* $\mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}} = \mathrm{Har}(\Pi, \mathscr{H}) \cap \mathscr{H}_{\mathrm{D}}$. *Then,*

$$\mathrm{Har}(\Pi, \mathscr{H}) \cap \left(\sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathscr{H}_{\mathrm{D}} \right) = \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}.$$

2. For every $\mathrm{D} \in \Omega(\mathrm{K}), \overline{\mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}} = \mathscr{H}_{\mathrm{D}}$.

Remark 3.4.10. For the first part of Lemma 3.4.9, we will prove a slightly different result using the Lie algebra $\mathfrak{k}_0$ of K. Because of our assumptions on the representations $\mathrm{D} \in \Omega(\mathrm{K})$, we get the following bijection:

$$\{\mathrm{D} \in \Omega(\mathrm{K}), \mathrm{Hom}_{\mathrm{K}}(\mathrm{D}, \Pi) \neq \{0\}\} \leftrightarrow \{\mathrm{D}^* \in \Omega(\mathrm{K}^*), \mathrm{Hom}_{\mathrm{K}^*}(\mathrm{D}^*, \Pi^*) \neq \{0\}\}. \tag{3.6}$$

The group K being simply connected, the finite dimensional representations of $\mathfrak{k}_0$ and K are in bijection.

We denote by $\Omega(\mathfrak{k}_0)$ the set of equivalence classes of simple representations of $\mathfrak{k}$. We first prove the following lemma.

Lemma 3.4.11. *Let* (ρ, V) *be a representation of* $\mathfrak{k}$ *and* W *be any* ρ*-stable subspace of* V*. Then,*

$$W \cap \left(\sum_{\mathrm{D}\in\Omega(\mathfrak{k})} V_{\mathrm{D}} \right) = \sum_{\mathrm{D}\in\Omega(\mathfrak{k})} W \cap V_{\mathrm{D}}.$$

Proof. Let x be an element of $\sum_{\mathrm{D}\in\Omega(\mathfrak{k})} V_{\mathrm{D}}$, and V_x the smallest ρ-stable subspace containing x. Obviously, the space V_x is contained in the sum of a finite number of simple spaces. Because $\mathfrak{k}$

is reductive, we get, according to a result of Chevalley that

$$V_x = \sum_{\mathrm{D}\in\Omega(\mathfrak{k})} V_x \cap V_{\mathrm{D}}.$$

By starting with $x \in W \cap \left(\sum_{\mathrm{D}\in\Omega(\mathfrak{k})} V_{\mathrm{D}}\right)$, we get that

$$W \cap \left(\sum_{\mathrm{D}\in\Omega(\mathfrak{k})} V_{\mathrm{D}}\right) \subseteq \sum_{\mathrm{D}\in\Omega(\mathfrak{k})} W \cap V_{\mathrm{D}},$$

and then, we get the equality (the other inclusion being obvious).

□

Proof of Lemma 3.4.9. 1. Obvious Using that $\mathrm{Har}(\Pi, \mathscr{H})$ is G-invariant and Equation (3.6).

2. We know that $\mathscr{H}_{\mathrm{D}} = \mathrm{E}_{\mathrm{D}}\mathscr{H}$ is closed. It's obvious to see that $\sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}$ is stable under E_{D}.

   Let $\Psi \in \mathscr{H}_{\mathrm{D}}$. Because $\sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}$ is dense under $\mathscr{H}$, there exists $\Psi_n \in \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}$ such that $\Psi = \lim_{n\to+\infty} \Psi_n$. Then,

$$\Psi = \mathrm{E}_{\mathrm{D}}(\Psi) = \mathrm{E}_{\mathrm{D}}(\lim_{n\to+\infty} \Psi_n) = \lim_{n\to+\infty} \mathrm{E}_{\mathrm{D}}(\Psi_n),$$

   where $\mathrm{E}_{\mathrm{D}}(\Psi_n) \in \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}$. Then, $\mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}$ is dense in $\mathscr{H}$.

□

Remark 3.4.12. The second statement of Lemma 3.4.1 can be given in a more general way. Let V be a subspace of $\mathscr{H}$ which is $\Pi(\mathrm{K})$-invariant and denote bt $V_{\mathrm{D}} = V \cap \mathscr{H}_{\mathrm{D}}$. If Π is permissible, then $\sum_{\mathrm{D}\in\Omega(\mathrm{K})} V_{\mathrm{D}}$ is dense in $\mathscr{H}$ and $\overline{V_{\mathrm{D}}} = \mathscr{H}_{\mathrm{D}}$. The proof is similar.

3.5 quasi-simple representations

We start with a definition.

Definition 3.5.1. The permissible Π is called quasi-simple if:

1. Π is permissible,
2. There exists an homomorphism χ of $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ into $\mathbb{C}$ such that $d\Pi_{\mathrm{G}}(z)\Psi = \chi(z)\Psi$ for every $z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ and $\Psi \in \mathrm{Gar}(\Pi, \mathscr{H})$.

Exercice 8. Let $(\Pi, \mathscr{H})$ be a quasi-simple representation of G and let $\chi : \mathrm{Z}(\mathscr{U}(\mathfrak{g})) \to \mathbb{C}$ be the corresponding character.Then, $d\Pi_{\mathrm{H}}(z)\Psi = \chi(z)\Psi$ for every $z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ and $\Psi \in \mathrm{Har}(\Pi, \mathscr{H})$.

The following result will play a fundamental role to prove that the trace of the operator $\Pi(\Psi)$ is well-defined for every $\Psi \in \mathscr{C}_c^{\infty}(\mathrm{G})$.

Proposition 3.5.2. *Let $(\Pi, \mathscr{H})$ be a quasi-simple representation of G. Then, for all $\mathrm{D} \in \Omega(\mathrm{K})$, we get that $\dim_{\mathbb{C}}(\mathscr{H}_{\mathrm{D}}) < \infty$. In particular, $\mathrm{Har}(\Pi, \mathscr{H}) = \mathscr{H}_{\mathrm{D}}$.*

Before proving this proposition, we are going to state two lemmas that will admit.

Lemma 3.5.3. *Let (ρ, V) be a representation of $\mathfrak{g}$ (and in particular, its restriction to $\mathfrak{k}$ gives a representation of $\mathfrak{k}$). Then, the space $\sum\limits_{\mathrm{D}\in\Omega(\mathrm{K})} V_{\mathrm{D}}$ is invariant under $\rho(\mathscr{U}(\mathfrak{g}))$.*

Lemma 3.5.4. *Let $\mathscr{I}$ be a left ideal in $\mathscr{U}(\mathfrak{k})$ (algebra generated by 1 and $\mathfrak{k}$ in $\mathscr{U}(\mathfrak{g})$). Assume that $\mathscr{U}(\mathfrak{k})/\mathscr{I}$ is finite dimensional and such that the natural representation of on $\mathscr{U}(\mathfrak{k})/\mathscr{I}$ is semi-simple. Then, the natural representation of $\mathfrak{k}$ on $\mathscr{U}(\mathfrak{g})/\mathscr{U}(\mathfrak{g})\mathscr{I}$ is quasi-simple. We denote by $\mathscr{B}^*$ the quotient $\mathscr{U}(\mathfrak{g})/\mathscr{U}(\mathfrak{g})\mathscr{I}$ and by $\mathscr{B}^*_{\mathrm{D}}$ the D-isotypic component $(\mathrm{D} \in \Omega(\mathrm{K}))$ under the action of $\mathfrak{k}$ on $\mathscr{B}^*$.*
Then, $\mathscr{B}^*_{\mathrm{D}}$ is a finite module over $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$.

Proof of the Proposition 3.5.2. Let Ψ_0 be an element of $\sum\limits_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}$. We denote by $\mathrm{U}_0 = d\Pi(\mathscr{U}(\mathfrak{g}))\Psi_0$ and $\mathrm{U} = \overline{d\Pi(\mathscr{U}(\mathfrak{g}))\Psi_0}$. According to Corollary 3.3.10, the space U is G-invariant. From Lemma 3.4.9, we get that:

$$\mathrm{U}_0 \subseteq \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}.$$

Then, we get:

$$U_0 = U_0 \cap \left(\sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}} \right) = \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{U}_0 \cap \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}},$$

where the second equality follows from Lemma 3.4.11. We denote by $\mathscr{I}$ the ideal of $\mathscr{U}(\mathfrak{k})$ given by:

$$\mathscr{I} = \{X \in \mathscr{U}(\mathfrak{k}), d\Pi(X)\Psi_0 = 0\}.$$

Exercice 9. Prove that the quotient $\mathscr{U}(\mathfrak{k})/\mathscr{I}$ is finite-dimensional and that the $\mathfrak{k}$-action on the quotient is semi-simple.

We keep the notations of Lemma 3.5.4. We denote by Π^* the natural projection of $\mathscr{U}(\mathfrak{g})$ on $\mathscr{U}(\mathfrak{g})^* = \mathscr{U}(\mathfrak{g})/\mathscr{U}(\mathfrak{g})\mathscr{I}$. We define the form

$$\alpha^* : \mathscr{U}(\mathfrak{g})^* \ni b^* = \Pi^*(b) \to d\Pi(b)\Psi_0 \in \mathrm{U}_0\,.$$

Obviously, the form α^* is well-defined. Indeed, if b_1 and b_2 are two elements of $\mathscr{U}(\mathfrak{g})$ such that $\Pi^*)(b_1) = Pi^*(b_2)$, there exists $i \in \mathscr{I}$ such that $b_1 = b_2 + i$. Then,

$$\begin{aligned}\alpha^*(\Pi^*(b_1)) &= d\Pi(b_1)\Psi_0 = d\Pi(b_2+i)\Psi_0 = d\Pi(b_2)\Psi_0 + d\Pi(i)\Psi_0 \\ &= d\Pi(b_2)\Psi_0 = \alpha^*(\Pi^*(b_2)).\end{aligned}$$

Moreover, for $a \in \mathscr{U}(\mathfrak{k})$ and $b \in \mathscr{U}(\mathfrak{g})^*$,

$$\alpha^*(\Pi^*(a)b^*) = \alpha(\Pi^*(a)\Pi^b) = \alpha(\Pi^*(ab))\, d\Pi(ab)\Psi_0 = d\Pi(a)(d\Pi(b)\Psi_0) = d\Pi(a)\alpha^*(b). \quad (3.7)$$

According to Lemma 3.5.4, we get that $\mathscr{B}^*_{\mathrm{D}}$ is a finite module over $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ for $\mathrm{D} \in \Omega(\mathrm{K})$. We clearly get that

$$\alpha(\mathscr{B}^*_{\mathrm{D}}) = \mathrm{U}_0 \cap \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}.$$

There exists elements $b^*_{1,\mathrm{D}}, \ldots, b^*_{n,\mathrm{D}} \in \mathscr{B}^*_{\mathrm{D}}$ such that

$$\mathscr{B}^*_{\mathrm{D}} = \sum_{i=1}^{n} \Pi^*(\mathrm{Z}(\mathscr{U}(\mathfrak{g})))b^*_{i,\mathrm{D}}. \quad (3.8)$$

According to Equation 3.7, we get:

$$\alpha^*(\mathscr{B}^*_{\mathrm{D}}) = \sum_{i=1}^{n} \alpha(\Pi^*(\mathrm{Z}(\mathscr{U}(\mathfrak{g})))b^*_{i,\mathrm{D}}) = \sum_{i=1}^{n} d\Pi(\Pi^*(\mathrm{Z}(\mathscr{U}(\mathfrak{g}))))\alpha(b^*_{i,\mathrm{D}}). \quad (3.9)$$

By assumptions, Π is quasi-simple. In particular, we get:

$$\mathrm{U}_0 \cap \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}} = \alpha^*(\mathscr{B}^*_{\mathrm{D}}) = \sum_{i=1}^{n} \chi(\mathrm{Z}(\mathscr{U}(\mathfrak{g}))\alpha^*(b^*_{i,\mathrm{D}}). \quad (3.10)$$

In particular, $\{\alpha^*(b_{i,D}), 1 \leq i \leq n\}$ spans the space $\mathrm{U}_0 \cap \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}}$, and this space is finite-dimensional. By definition of U_{D}, we get:

$$\mathrm{U}_0 \cap \mathrm{Har}(\Pi, \mathscr{H})_{\mathrm{D}} = \mathrm{U}_0 \cap \mathrm{Har}(\Pi, \mathscr{H}) \cap \mathscr{H}_{\mathrm{D}} = \mathrm{U}_0 \cap \mathscr{H}_{\mathrm{D}} = \mathrm{U}_0 \cap \mathrm{U}_{\mathrm{D}}\,. \quad (3.11)$$

Because $\mathrm{U}_0 = \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{U}_0 \cap \mathrm{U}_{\mathrm{D}}$ is dense in U, we get that $\mathrm{U}_{\mathrm{D}} = \overline{\mathrm{U}_0 \cap \mathrm{U}_{\mathrm{D}}} = \mathrm{U}_0 \cap \mathrm{U}_{\mathrm{D}}$ because the space is finite dimensional.

□

We get directly the following corollary.

Corollary 3.5.5. *1. Let* $(\Pi, \mathscr{H})$ *be an irreducible quasi-simple representation of* G. *Then, for every* $\mathrm{D} \in \Omega(\mathrm{K})$, *we have* $\dim_{\mathbb{C}} \mathscr{H}_{\mathrm{D}} < \infty$.

2. *Every element in* $\sum_{D \in \Omega(K)} \mathscr{H}_D$ *is analytic.*

Assume now that $\mathscr{H}$ is a complex Hilbert space whose scalar product is denoted by $\langle \cdot, \cdot \rangle$.

Definition 3.5.6. The representation $(\Pi, \mathscr{H})$ is said to be unitary if for every $u, v \in \mathscr{H}$, we get:

$$\langle \Pi(g)u, \Pi(g)v \rangle = \langle u, v \rangle \qquad (g \in G). \tag{3.12}$$

We will admit the following result of Mautner.

Theorem 3.5.7. *Every irreducible unitary representation* $(\Pi, \mathscr{H})$ *is quasi-simple.*

Chapter 4

Global character of an irreducible quasi-simple representation

The goal of this section is to prove the following theorem.

Theorem 4.0.1. *Let* $(\Pi, \mathscr{H})$ *be an irreducible quasi-simple representation of* G. *For every* $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$*, the operator*

$$\Pi(\Psi) = \int_{\mathrm{G}} \Psi(g)\Pi(g)dg \tag{4.1}$$

is of trace class, and the corresponding map:

$$\Theta_\Pi : \mathscr{C}_c^\infty(\mathrm{G}) \ni \Psi \to \mathrm{tr}(\Pi(\Psi)) \in \mathbb{C} \tag{4.2}$$

is a distribution (in the sense of Laurent Schwarz).
Moreover, let $(\Pi_1, \mathscr{H}_1)$ *and* $(\Pi_2, \mathscr{H}_2)$ *be two irreducible representations of* G. *The representations* $\Pi_!$ *and* Π_2 *are infinitesimally equivalent if and only if* $\Theta_{\Pi_1} = \Theta_{\Pi_2}$.

Through this section., $\mathscr{H}$ will be an Hilbert space over $\mathbb{C}$ and $\langle\cdot,\cdot\rangle$ the corresponding scalar product. For an orthonormal basis $\{\Psi_\alpha\}_{\alpha\in J}$, we get:

1. For every $x \in \mathscr{H}$,

$$x = \sum_{\alpha\in J} \langle x, \Psi_\alpha\rangle \Psi_\alpha, \tag{4.3}$$

2. For every $x, y \in \mathscr{H}$,

$$\langle x, y\rangle = \sum_{\alpha\in J} \langle x, \Psi_\alpha\rangle \langle \Psi_\alpha, y\rangle . \tag{4.4}$$

   In particular, $||x||^2 = \langle x, x\rangle = \sum_{\alpha\in J} | \langle x, \Psi_\alpha\rangle |$.

4.1 Trace class operators

Definition 4.1.1. Let A $\in \mathscr{B}(\mathscr{H})$ be a bounded operator on $\mathscr{H}$ and $\{\Psi_\alpha\}_{\alpha\in J}$ be an orthonormal basis of $\mathscr{H}$. We say that A is of trace class if for every such basis, the series

$$\sum_{\alpha\in J} \langle A\Psi_\alpha, \Psi_\alpha\rangle \tag{4.5}$$

converges to a sum which is independant of the choice of the basis. We will denote this sum by $\mathrm{tr}(A)$.

We. prove the following lemma.

Lemma 4.1.2. *Let $\{\Psi_\alpha\}_{\alpha\in J}$ be an orthonormal basis of $\mathscr{H}$ and T be a bounded operator such that*

$$\sum_{\alpha,\beta\in J} |\left\langle T\Psi_\alpha, \Psi_\beta\right\rangle| < \infty. \tag{4.6}$$

Then, is A and B are any bounded operators on $\mathscr{H}$, the operators ATB, BAT and TBA are of trace class and $\mathrm{tr}(ATB) = \mathrm{tr}(BAT) = \mathrm{tr}(TBA)$. *In particular, for any regular operator A, ATA^{-1} is of trace class and* $\mathrm{tr}(ATA^{-1}) = \mathrm{tr}(T)$.

Proof. Fix an orthonormal basis $\{\Psi_\alpha\}_{\alpha\in J}$ of $\mathscr{H}$. Then,

$$\begin{aligned}
\sum_{\alpha\in J} \langle ATB\Psi_\alpha, \Psi_\alpha\rangle &= \sum_{\alpha\in J}\left\langle AT\left(\sum_{\beta\in J}\left\langle B\Psi_\alpha, \Psi_\beta\right\rangle \Psi_\beta\right), \Psi_\alpha\right\rangle \\
&= \sum_{\alpha\in J}\sum_{\beta\in J}\left\langle B\Psi_\alpha, \Psi_\beta\right\rangle\left\langle A\left(\sum_{\gamma\in J}\left\langle T\Psi_\beta, \Psi_\gamma\right\rangle \Psi_\gamma\right), \Psi_\alpha\right\rangle \\
&= \sum_{\alpha,\beta,\gamma\in J}\left\langle B\Psi_\alpha, \Psi_\beta\right\rangle\left\langle T\Psi_\beta, \Psi_\gamma\right\rangle\left\langle A\Psi_\gamma, \Psi_\alpha\right\rangle \\
&= \sum_{\beta,\gamma\in J}\left\langle T\Psi_\beta, \Psi_\gamma\right\rangle\left(\sum_{\alpha\in J}\left\langle A\Psi_\gamma, \Psi_\alpha\right\rangle\left\langle B\Psi_\alpha, \Psi_\beta\right\rangle\right)
\end{aligned}$$

Obviously, using Holder's inequality, we get:

$$\begin{aligned}
\left|\sum_{\alpha\in J}\left\langle A\Psi_\gamma, \Psi_\alpha\right\rangle\left\langle B\Psi_\alpha, \Psi_\beta\right\rangle\right| &\le \sum_{\alpha\in J}\left|\left\langle A\Psi_\gamma, \Psi_\alpha\right\rangle\left\langle B\Psi_\alpha, \Psi_\beta\right\rangle\right| \\
&\le \left(\sum_{\beta\in J}\left|\left\langle B\Psi_\alpha, \Psi_\beta\right\rangle\right|^2\right)^{\frac{1}{2}}\left(\sum_{\alpha\in J}\left|\left\langle B\Psi_\gamma, \Psi_\alpha\right\rangle\right|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Using Equation (4.4),

$$\sum_{\alpha\in J}\left|\left\langle B\Psi_\gamma, \Psi_\alpha\right\rangle\right|^2 = \left\|A\Psi_\gamma\right\|^2 \le N(A)^2\left\|\Psi_\gamma\right\|^2 = N(A)^2, \tag{4.7}$$

and because B is a bounded operator,

$$\sum_{\beta\in J}\left|\left\langle B\Psi_\alpha,\Psi_\beta\right\rangle\right|^2=\sum_{\beta\in J}\left|\left\langle \Psi_\alpha,B^*\Psi_\beta\right\rangle\right|^2=\left\|B^*\Psi_\beta\right|^2\leq N(B^*)^2=N(B)^2. \tag{4.8}$$

Then,

$$\left|\sum_{\alpha,\beta,\gamma\in J}\left\langle B\Psi_\alpha,\Psi_\beta\right\rangle\left\langle T\Psi_\beta,\Psi_\gamma\right\rangle\left\langle A\Psi_\gamma,\Psi_\alpha\right\rangle\right|\leq N(A)N(B^*)\sum_{\beta,\gamma\in J}\left\|\left\langle T\Psi_\beta,\Psi_\gamma\right\rangle\right\|<\infty. \tag{4.9}$$

In particular,

$$\sum_{\alpha\in J}\langle ATB\Psi_\alpha,\Psi_\alpha\rangle=\sum_{\alpha\in J}\langle TAB\Psi_\alpha,\Psi_\alpha\rangle=\sum_{\alpha\in J}\langle BAT\Psi_\alpha,\Psi_\alpha\rangle<\infty. \tag{4.10}$$

If we take another orthonormal basis $\{\beta_\alpha\}_{\alpha\in J}$, there exists a unitary operator $U\in \mathrm{U}(\mathscr{H})$ such that $\beta_\alpha=U\Psi_\alpha$. Then,

$$\begin{aligned}\sum_{\alpha\in J}\langle ATB\beta_\alpha,\beta_\alpha\rangle &= \sum_{\alpha\in J}\langle ATBU\Psi_\alpha,U\Psi_\alpha\rangle\\ &= \sum_{\alpha\in J}\left\langle (U^{-1}A)T(BU)\Psi_\alpha,\Psi_\alpha\right\rangle=\sum_{\alpha\in J}\left\langle (BU)(U^{-1}A)T\Psi_\alpha,\Psi_\alpha\right\rangle\\ &= \sum_{\alpha\in J}\langle BAT\Psi_\alpha,\Psi_\alpha\rangle=\sum_{\alpha\in J}\langle ATB\Psi_\alpha,\Psi_\alpha\rangle\end{aligned}$$

In particular, the value of the sum $\sum_{\alpha\in J}\langle ATB\Psi_\alpha,\Psi_\alpha\rangle$ is independant of the choice of the basis. In particular, ATB, BAT and TBA are trace class operators and we get $\mathrm{tr}(ATB)=\mathrm{tr}(BAT)=\mathrm{tr}(TBA)$.

□

Remark 4.1.3. In this chapter, we want to prove that $\Pi(\Psi),\Psi\in\mathscr{C}_c^\infty(\mathrm{G})$ is of trace class. We are going to prove something stronger, i.e. that $\Pi(\Psi)$ satisfies the inequality given in Equation (4.6).

4.2 Three fundamental theorems

In this section, we are going to state three theorems that will be useful for us in the rest of the chapter.

Theorem 4.2.1. *Let* $(\Pi,\mathscr{H})$ *be a quasi-simple irreducible representation of* G*. There exists an integer* $N\in\mathbb{Z}^+$ *such that*

$$\dim_{\mathbb{C}}(\mathscr{H}_{\mathrm{D}})\leq Nd(\mathrm{D})^2 \qquad (D\in\Omega(\mathrm{K})), \tag{4.11}$$

where $d(\mathrm{D})$ is the dimension of the finite dimensional representation D.

Theorem 4.2.2. *There exists an element $z \in \mathscr{U}(\mathfrak{k})$ such that:*

$$\sum_{\mathrm{D}\in\Omega(\mathrm{K})} \|E_{\mathrm{D}}\Psi\| \leq \|\Pi(z)\Psi\| \tag{4.12}$$

for every $\Psi \in \mathrm{Diff}(\Pi, \mathrm{H}) = \left\{\Psi \in \mathscr{H}, \lim\limits_{t\to 0} \frac{\Pi(\exp(tX))\Psi - \Psi}{t} \text{ exists}\right\}$.
Moreover, the series $\sum_{\mathrm{D}\in\Omega(\mathrm{K})} E_{\mathrm{D}}\Psi$ *converges to* Ψ.

Theorem 4.2.3. *Let's denote by K' the commutator subgroup of K and by $\mathfrak{k}'_0$ and $\mathfrak{k}'$ the corresponding Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. There exists $z_0 \in \mathrm{Z}(\mathscr{U}(\mathfrak{k}')$ such that:*

$$D(z_0) = d(\mathrm{D})^2\, \mathrm{D}(1) \tag{4.13}$$

for every $\mathrm{D} \in \Omega(\mathrm{K})$.

4.3 The operators $\Pi(\Psi), \Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, are of trace class

Let $(\Pi, \mathscr{H})$ be a quasi-simple irreducible representation of G. We denote by $\mathscr{D}_0$ the subspace of $\mathscr{B}(\mathscr{H})$ generated by the operators $\Pi(\Psi), \Psi \in \mathscr{C}_c^\infty(\mathrm{G})$ and by $\mathscr{D}$ its closure in $\mathscr{B}(\mathscr{H})$. We denote by l and r the maps:

$$l : \mathrm{G} \to \mathscr{B}(\mathscr{D}) \qquad r : \mathrm{G} \to \mathscr{B}(\mathscr{D}) \tag{4.14}$$

given by $l(g)(A) = \Pi(g)A, r(g)A$, $g \in \mathrm{G}, A \in \mathscr{D}$.
For all $f \in \mathscr{C}_c^\infty(\mathrm{G})$ and $g \in \mathrm{G}$, we get:

$$\begin{aligned} \Pi(g)\circ\Pi(f) &= \Pi(g)\circ\left(\int_{\mathrm{G}} f(h)\Pi(h)dh\right) = \int_{\mathrm{G}} f(h)\Pi(gh)dh = \int_{\mathrm{G}} fg^{-1}(h)\Pi(h)dh \\ &= \int_{\mathrm{G}} L_{g^{-1}}f(h)\Pi(h)dh = \Pi(f\circ L_{g^{-1}}) \end{aligned}$$

and because $L_{g^{-1}}$ is a bijection, $f \circ L_{g^{-1}} \in \mathscr{C}_c^\infty(\mathrm{G})$. It follows that $\mathscr{D}_0$ is a G-invariant subspace for l and r.

Exercice 10. Prove that l and r are continuous permissible representations of G.

For all $\mathrm{D} \in \Omega(\mathrm{K})$, we denote by $\mathscr{H}_{\mathrm{D}}$, $\mathscr{D}_{\mathrm{D}}$ and $\mathscr{D}_{\mathrm{D}}$ the D-isotypic component corresponding to Π, l and r. We denote by:

$$\mathrm{E_D} : \mathscr{H} \to \mathscr{H}_{\mathrm{D}} \qquad \mathrm{P_D} : \mathscr{D} \to \mathscr{D}_{\mathrm{D}} \qquad .\mathrm{Q_D} : \mathscr{D} \to \mathscr{D}_{\mathrm{D}} \tag{4.15}$$

the corresponding projections. Obviously, for every $A \in \mathscr{D}$, we get:

$$\begin{aligned} \mathrm{E_D} \circ A &= \left(\int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}^*}(u^*)} \Pi^*(u^*) du^* \right) \circ A = \int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}^*}(u^*)} \Pi^*(u^*) \circ A du^* \\ &= \int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}^*}(u^*)} l^*(u^*)(A) du^* = \mathrm{P_D}(A). \end{aligned}$$

Similarly,

$$\begin{aligned} \mathrm{P_D}(A) &= \int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}^*}(u^*)} r^*(u^*)(A) du^* = \int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}^*}(u^*)} A \circ \Pi^*(u^{*-1}) du^* \\ &= \int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}^*}(u^*)^{-1}} A \circ \Pi^*(u^*) du^* = A \circ \left(\int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}'^*}(u^*)^{-1}} \Pi^*(u^*) du^* \right) \\ &= A \circ \mathrm{E_{D'}} \end{aligned}$$

where D' is. the contragredient representation of D. Because of the definition of K, we get $[\mathrm{D}] = [\mathrm{D}']$.

|let's denote by λ and ρ the left and right regular representations of G on the space $\mathrm{L}^2(\mathrm{G}, dg)$. Obviously, $\mathscr{C}_c^\infty(\mathrm{G}) \subseteq \mathrm{L}^2(\mathrm{G}, dg)$ and the elements of $\mathscr{C}_c^\infty(\mathrm{G})$ are in $\mathrm{Diff}(L^2(\mathrm{G}, dg), \lambda)$ and $\mathrm{Diff}(L^2(\mathrm{G}, dg), \rho)$. Moreover, $\mathscr{C}_c^\infty(\mathrm{G})$ is stable under λ and ρ, and for every $g, h \in \mathrm{G}$ and $f \in \mathscr{C}_c^\infty(\mathrm{G})$, we get:

$$\begin{aligned} l(x) \circ r(y)(\Pi(f)) &= l(x) \circ \left(\int_{\mathrm{G}} f(g) \Pi(g) \circ \Pi(y^{-1}) dg \right) = \int_{\mathrm{G}} f(g) \Pi(x) \circ \pi(g) \circ \Pi(y^{-1}) dg \\ &= \int_{\mathrm{G}} f(x^{-1} g y) \Pi(g) dg = \Pi(\lambda(x) \rho(y) f) \end{aligned}$$

It follows easily that $l(a) r(b) \Pi(f) = \Pi(\lambda(a) \rho(b) f)$ for every $a, b \in \mathscr{U}(\mathfrak{g})$.
To simplify the notations, we still denote by ρ the differential of ρ actiong on $\mathrm{Diff}(L^2(\mathrm{G}, dg), \rho)$ or $\mathscr{C}_c^\infty(\mathrm{G})$.

Proposition 4.3.1. *For all $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, the operator $\Pi(f)$ is of trace class.*

Proof. We define a map

$$\Phi : \mathrm{G} \times \mathrm{G} \ni (x, y) \to \left(A \to \Phi(x, y)(A) = \Pi(x) \circ A \circ \Pi(y^{-1}) \right) \in \mathscr{B}(\mathscr{D}). \qquad (4.16)$$

The group $\mathrm{G} \times \mathrm{G}$ is semi-simple, the analytic subgroups corresponding to $\mathrm{Fix}(\mathfrak{g}_0 \oplus \mathfrak{g}_0, \theta \oplus \theta)$ is $\mathrm{K} \times \mathrm{K}$, and the representation Φ is semi-simple. Indeed, for every $(x, y) \in (\mathrm{Z}(\mathrm{G}) \cap \mathrm{D})^2$, we get:

$$\Phi(x, y) A = \Pi(x) \circ A \circ \Pi(y^{-1}) = \xi_\Pi(x) \xi_\Pi(y^{-1}) A = \xi_\Pi(x y^{-1}) A. \qquad (4.17)$$

1. We first assume that the spaces $\mathscr{H}_\mathrm{D}$ are mutually orthogonal ($\mathrm{D} \in \Omega(\mathrm{K})$).

   We choose an orthonormal basis of each $\mathscr{H}_\mathrm{D}$ and then, we get a basis $\{\Psi_\alpha\}_{\alpha \in J}$. For all

$\mathrm{D} \in \Omega(\mathrm{K})$, we denote by J^{D} the finite set of J such that $\{\Psi_\alpha\}_{\alpha \in J^{\mathrm{D}}}$ is a basis of $\mathscr{H}_{\mathrm{D}}$. Then,

$$\begin{aligned}
\sum_{\alpha,\beta \in J} |\langle \Pi(f)\Psi_\alpha, \Psi_\beta \rangle| &= \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} \sum_{\alpha \in J^{\mathrm{D}_1}} \sum_{\beta \in J^{\mathrm{D}_2}} |\langle \Pi(f)\Psi_\alpha, \Psi_\beta \rangle| \\
&= \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} \sum_{\alpha \in J^{\mathrm{D}_1}} \sum_{\beta \in J^{\mathrm{D}_2}} |\langle E_{\mathrm{D}_2} \circ \Pi(f) \circ E_{\mathrm{D}_1}(\Psi_\alpha), \Psi_\beta \rangle| \\
&\leq \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} \sum_{\alpha \in J^{\mathrm{D}_1}} \sum_{\beta \in J^{\mathrm{D}_2}} \|\Psi_\beta\| \|E_{\mathrm{D}_2} \circ \Pi(f) \circ E_{\mathrm{D}_1}(\Psi_\alpha)\| \\
&\leq \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} \sum_{\alpha \in J^{\mathrm{D}_1}} \sum_{\beta \in J^{\mathrm{D}_2}} \|\Psi_\beta\| \|\Psi_\alpha)\| N(E_{\mathrm{D}_2} \circ \Pi(f) \circ E_{\mathrm{D}_1}) \\
&= \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} \dim_{\mathbb{C}} \mathscr{H}_{\mathrm{D}_1} \dim_{\mathbb{C}} \mathscr{H}_{\mathrm{D}_2} N(E_{\mathrm{D}_2} \circ \Pi(f) \circ E_{\mathrm{D}_1}) \\
&\leq N \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} d(\mathrm{D}_1)^2 d(\mathrm{D}_2)^2 N(E_{\mathrm{D}_2} \circ \Pi(f) \circ E_{\mathrm{D}_1})
\end{aligned}$$

Obviously, the projections on the $\mathrm{D}_1 \times \mathrm{D}_2$-isotypic component for Φ are of the form $P_{\mathrm{D}_1} Q_{\mathrm{D}_2}$. By applying Theorem 4.25, we get:

$$\sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} N\left(P_{\mathrm{D}_1} Q_{\mathrm{D}_2} \Pi(\lambda(z_0)\rho(z_0)f\right) \leq N(\Phi(z)\Pi(\lambda(z_0)\rho(z_0)f) < \infty, \tag{4.18}$$

where z_0 is the element defined in Theorem 4.2.3. Using the previous discussion, we get:

$$\begin{aligned}
\sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} N\left(P_{\mathrm{D}_1} Q_{\mathrm{D}_2} \Pi(\lambda(z_0)\rho(z_0)f)\right) &= \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} N\left(P_{\mathrm{D}_1} Q_{\mathrm{D}_2} l(z_0) r(z_0) \Pi(f)\right) \\
&= \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} d(\mathrm{D}_1)^2 d(\mathrm{D}_2)^2 N(P_{\mathrm{D}_1} Q_{\mathrm{D}_2} \Pi(f)) \\
&= \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} d(\mathrm{D}_1)^2 d(\mathrm{D}_2)^2 N(E_{\mathrm{D}_1} \circ \Pi(f) \circ E_{\mathrm{D}_2})
\end{aligned}$$

Finally,

$$\sum_{\alpha,\beta \in J} |\langle \Pi(f)\Psi_\alpha, \Psi_\beta \rangle| \leq N \sum_{\mathrm{D}_1,\mathrm{D}_2 \in \Omega(\mathrm{K})} d(\mathrm{D}_1)^2 d(\mathrm{D}_2)^2 N(E_{\mathrm{D}_1} \circ \Pi(f) \circ E_{\mathrm{D}_2}) < \infty. \tag{4.19}$$

Then, $\Pi(f)$ is of trace class.

2. We now remove the assumption of orthogonality of the spaces $\mathscr{H}_\alpha$.

   As before, we denote by $\mathrm{K}^* = \mathrm{K} / \mathrm{Z(G)} \cap \mathrm{D}$ and by Π^* the corresponding representation of K^* on $\mathscr{H}$. Because K^* is compact, there exists an operator $U \in \mathrm{U}(\mathscr{H})$ such that $U\Pi^*(u*)U^{-1}$ is unitary, $u* \in \mathrm{K}^*$. We denote by Π' the map

$$\Pi' : \mathrm{G} \ni g \to \Pi'(g) = U\Pi(b)U^{-1} \in \mathscr{B}(\mathscr{H}). \tag{4.20}$$

The D-isotypic components $\mathscr{H}'_{\mathrm{D}}$ for $(\Pi', \mathscr{H})$ are orthogonal. By applying the first part of the proof to $\Pi'(f)$, we get that $\Pi'(f)$ is a trace class operator. Using Lemma 4.1.2 and that $\Pi'(f) = U\Pi(f)U^{-1}$, we get:

$$\mathrm{tr}(\Pi'(f)) = \mathrm{tr}(U\Pi(f)U^{-1}) = \mathrm{tr}(\Pi(f)). \tag{4.21}$$

Finally, $\Pi(f)$ is of trace class and it concludes the proof.

□

Lemma 4.3.2. *For every $f \in \mathscr{C}_c^\infty(\mathrm{G})$ and $a \in \mathrm{G}$, we get:*

$$\mathrm{tr}(\Pi(f^a)) = \mathrm{tr}(\Pi(f)) \tag{4.22}$$

where $f^a(x) = f(axa^{-1}), x \in \mathrm{G}$.

Proof. For $a \in \mathrm{G}$ and $f \in \mathscr{C}_c^\infty(\mathrm{G})$, we get:

$$\begin{aligned}\mathrm{tr}(\Pi(f^a)) &= \mathrm{tr}\int_{\mathrm{G}} f^a(g)\Pi(g)dg = \mathrm{tr}\int_{\mathrm{G}} f(aga^{-1})\Pi(g)dg = \mathrm{tr}\int_{\mathrm{G}} f(g)\Pi(a^{-1}ga)dg \\ &= \mathrm{tr}\left(\Pi(a^{-1}) \circ \left(\int_{\mathrm{G}} f(g)\Pi(g)dg\right) \circ \Pi(a)\right) = \mathrm{tr}(\Pi(f))\end{aligned}$$

□

4.4 The map Θ_Π is a distribution

4.4.1 A general result about distributions

We will state this result for $\mathrm{G} = \mathbb{R}^n$. The generalization ofr a general Lie group is quite straightforward.

Definition 4.4.1. A distribution T on $\mathbb{R}^n$ is a linear form on $\mathscr{C}_c^\infty(\mathbb{R}^n)$ such that for every compact set $\mathrm{K} \subseteq \mathbb{R}^n$, there exists a constant c_{K} and an integer n_{K} such that

$$|T(\Psi)| \leq c_{\mathrm{K}} \sum_{|\alpha| \leq n_{\mathrm{K}}} \sup N(\partial^\alpha \Psi) \tag{4.23}$$

where $\Psi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$.

The following theorem parametrizing distributions will be useful for us in the next section.

Theorem 4.4.2. *A linear form T on $\mathscr{C}_c^\infty(\mathbb{R}^n)$ is a distribution if and only if $T(\phi_j) \to 0$ when $j \to +\infty$ for every sequence ϕ_j of $\mathscr{C}_c^\infty(\mathbb{R}^n)$ converging to 0 (in the sense that $\sup(\partial^\alpha \phi_j) \to 0$ for every fixed α and $\mathrm{supp}(\phi_j) \subseteq \mathrm{K}$ for every j (and K is a fixed compact set in $\mathbb{R}^n$).*

Proof. See [16, Page 35, Theorem 2.1.4].

□

4.4.2 Proof of the result

Proposition 4.4.3. *The map* $\Theta_\Pi : \mathscr{C}_c^\infty(\mathrm{G}) \ni \Psi \to \mathrm{tr}(\Pi(\Psi)) \in \mathbb{C}$ *is a distribution.*

Proof. According to Theorem 4.4.2, what we are going to prove is that for any compact. set K of G and a sequence $\{f_n\}_{n\in\mathbb{Z}_*^+}$ of $\mathscr{C}_c^\infty(\mathrm{G})$ vanished outside $|K$ such that $\lambda(b)f_n \to 0$ uniformly on $|K$ for every $b \in \mathscr{U}(\mathfrak{g})$, then, $\lim_{n\to+\infty} |\Theta_\Pi(f_n)| = 0$.
Using the notations introduced previously, we get:

$$\Theta_\Pi(\Psi) = \mathrm{tr}(\Pi(\Psi)) = \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{tr}(E_\mathrm{D}\Pi(\Psi)E_\mathrm{D}). \tag{4.24}$$

Then,

$$\begin{aligned} |\Theta_\Pi(\Psi)| &\leq \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \sum_{\alpha\in J^\mathrm{D}} |\langle \Psi_\alpha, E_\mathrm{D}\Pi(\Psi)\Psi_\alpha\rangle| \leq \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \sum_{\alpha\in J^\mathrm{D}} ||\Psi_\alpha|| ||E_\mathrm{D}\Pi(\Psi)\Psi_\alpha|| \\ &\leq \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \dim_\mathbb{C}(\mathscr{H}_\mathrm{D}) N(E_\mathrm{D}\circ\Pi(\Psi)) \leq N \sum_{\mathrm{D}\in\Omega(\mathrm{K})} d(\mathrm{D})^2 N(E_\mathrm{D}\circ\Pi(\Psi)) \\ &= N \sum_{\mathrm{D}\in\Omega(\mathrm{K})} N(E_\mathrm{D}\circ\Pi(\lambda(z_0)\Psi)) \leq NN(l(z)l(z)\Pi(\lambda(z_0)\Psi) \qquad (Theorem \\ &= NN(\Pi(\lambda(zz_0)\Psi)) \end{aligned}$$

Let $\{f_n\}_n$ and C as before. In particular, $\lambda(b)f_n \to)$ uniformly. In particular,

$$N(\Pi(\lambda(b)f_n)) \leq \int_\mathrm{G} |\lambda(b)f_n(x)| N(\Pi(x))dx, \tag{4.25}$$

and then $|\Theta_\Pi(f_n)| \to 0, n \to +\infty$.

□

Remark 4.4.4. For a bounded operator B on the space $\mathscr{H}$, we say that B is of Hilbert-Schmidt class if $B * B$ is of trace class.
We can prove the following result: Let $(\Pi, \mathscr{H})$ be a quasi-simple irreducible representation of G on an Hilbert space $\mathscr{H}$ and f be a measurable and square integrable function on G which vanished outside a compact set. Them, the operator $\int_\mathrm{G} f(g)\Pi(g)dg$ is an Hilbert-Schmidt operator.

4.5 Fundamental property of characters

We are going to see in this section that for that the irreducible unitary representations are completely parametrised by their characters.

Definition 4.5.1. Let $(\Pi_1, \mathscr{H}_1)$ and $(\Pi_2, \mathscr{H}_2)$ be two permissible representations of G. Let

$$\mathscr{H}_1^0 = \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi_1, \mathscr{H}_1) \cap \mathscr{H}_1^{\mathrm{D}} \qquad \mathscr{H}_2^0 = \sum_{\mathrm{D}\in\Omega(\mathrm{K})} \mathrm{Har}(\Pi_2, \mathscr{H}_2) \cap \mathscr{H}_2^{\mathrm{D}}. \tag{4.26}$$

The representations Π_1 and Π_2 are infinitesimally equivalent if there exists a $1-1$ linear mapping $\alpha : \mathscr{H}_1^0 \to \mathscr{H}_2^0$ such that:

$$\Pi_2(b)\alpha(\Psi) = \alpha(\Pi_1(b)\Psi) \qquad (b \in \mathscr{U}(\mathfrak{k}), \Psi \in \mathscr{H}_1^0). \tag{4.27}$$

Exercice 11. Prove that if Π_1 and Π_2 are equivalent, then Π_1 and Π_2 are infinitesimally equivalent.

Remark 4.5.2. The converse is not true in general. We will see later that those two notions are equivalent for irreducible unitary representations.

Let $(\Pi, \mathscr{H})$ be an irreducible quasi-simple representation and $\mathrm{D} \in \Omega(\mathrm{K})$. We denote by $\mathscr{H}_{\mathrm{D}}$ the D-isotypic component and by $\{\Psi_\alpha\}_{\alpha\in J^{\mathrm{D}}}$ an orthonormal basis of $\mathscr{H}_{\mathrm{D}}$ (as in the proof of Proposition 4.3.1). We denote by Φ_{D}^{Π} the following function:

$$\Phi_{\mathrm{D}}^{\Pi} = \mathrm{tr}(E_{\mathrm{D}} \circ \Pi(g) \circ E_{\mathrm{D}}) = \sum_{\alpha\in J^{\mathrm{D}}} \langle E_{\mathrm{D}} \circ \Pi(g) \circ E_{\mathrm{D}}(\Psi_\alpha), \Psi_\alpha \rangle . \tag{4.28}$$

In particular, $\Phi_{\mathrm{D}}^{\Pi} = \mathrm{K}_{\mathrm{D}}\chi_{\mathrm{D}}$, where $\mathrm{K}_{\mathrm{D}} = |J^{\mathrm{D}}|/d(\mathrm{D})$. We admit the following theorem.

Theorem 4.5.3. *Let $\Pi_1, \dots, \Pi_r$ be a finite set of quasi-simple irreducible representations of G on the spaces $\mathscr{H}_1, \dots, \mathscr{H}_r$. Suppose no two of them are infinitesimally equivalent. Then, all the functions $\Phi_{\mathrm{D}_1}^{\Pi_1}, \dots, \Phi_{\mathrm{D}_r}^{\Pi_r}, D_i \in \Omega(\mathrm{K})$, are linearly independant.*

We get the following corollary.

Corollary 4.5.4. *Let Π_1 and Π_2 be two infinitesimally equivalent irreducible quasi-simple representations of G, then $\Theta_{\Pi_1} = \Theta_{\Pi_2}$.*

Proof. □

We now prove the following theorem.

Theorem 4.5.5. *Let $\Pi_1, \dots, \Pi_r$ be a finite set of quasi-simple irreducible representations of G on the spaces $\mathscr{H}_1, \dots, \mathscr{H}_r$. Suppose no two of them are infinitesimally equivalent. Then, all the functions $\Theta_{\Pi_1}, \dots, \Theta_{\Pi_r}, D_i \in \Omega(\mathrm{K})$ are linearly independant.*

Proof. Assume that $\sum_{i=1}^{q} c_i\Theta_{\Pi_i} = 0$ where $c_1 \neq 0, c_i \in \mathbb{C}$ and $q \leq r$.We know that all the $\Pi's$ are permissible. In particular, there exists characters $\eta_i : \mathrm{Z(G)} \cap \mathrm{D} \to \mathbb{C}$ such that:

$$\Pi_i(\gamma) = \ni_i (\gamma)\Pi(1) \qquad (1 \leq i \leq s). \tag{4.29}$$

For $\gamma \in \mathrm{Z(G)} \cap \mathrm{D}$, we denote by f_γ the function on G given by:

$$f_\gamma(g) = f(\gamma^{-1}g) \qquad (g \in \mathrm{G}). \tag{4.30}$$

Obviously, because $L_{\gamma^{-1}}$ is a bijection, the function f_γ is in $\mathscr{C}_c^\infty(\mathrm{G})$. For all $i \in [|1, q|]$, we get:

$$\begin{aligned}
\Theta_{\Pi_i}(f_\gamma) &= \operatorname{tr} \int_{\mathrm{G}} f_\gamma(g)\Pi_i(g)dg = \operatorname{tr} \int_{\mathrm{G}} f(\gamma^{-1}g)\Pi_i(g)dg \\
&= \operatorname{tr} \int_{\mathrm{G}} f(g)\Pi_i(\gamma g)dg = \operatorname{tr} \int_{\mathrm{G}} f(g)\Pi_i(\gamma)\Pi_i(g)dg \\
&= \eta_i(\gamma) \operatorname{tr} \int_{\mathrm{G}} f(g)\Pi_i(g)dg = \eta_i(\gamma)\Theta_{\Pi_i}(f)
\end{aligned}$$

In particular, we get:

$$\sum_{i=1}^{q} c_i\eta_i(\gamma)\Theta_{\Pi_i} = 0 \qquad (\gamma \in \mathrm{D} \cap \mathrm{Z(G)}). \tag{4.31}$$

Let $J = \{i \in [|2, q|], \eta_1 = \eta_i\}$. Then, using Equation (4.31),

$$\eta_1(\gamma) \sum_{\substack{i=1 \\ i \in J}}^{q} c_i\eta_i(\gamma)\Theta_{\Pi_i} + \sum_{\substack{i=1 \\ i \notin J}}^{q} \in c_i\eta_i(\gamma)\eta_i(\gamma)\Theta_{\Pi_i} \qquad (\gamma \in \mathrm{Z(G)} \cap \mathrm{D}). \tag{4.32}$$

To simplify the notations, we will assume that $J = \{1, 2, \ldots, s\}$, i.e.

$$\eta_1(\gamma) \sum_{i=1}^{s} c_i\eta_i(\gamma)\Theta_{\Pi_i} + \sum_{i=s+1}^{q} \in c_i\eta_i(\gamma)\eta_i(\gamma)\Theta_{\Pi_i} \qquad (\gamma \in \mathrm{Z(G)} \cap \mathrm{D}). \tag{4.33}$$

Exercice 12. Prove that Equation (4.33) implies that

$$\sum_{i=1}^{s} c_i\Theta_{\Pi_i} = 0. \tag{4.34}$$

As recalled in [4], we have $\mathrm{G} = \mathrm{K\,S}$, with $\mathrm{K} = [\mathrm{K}, \mathrm{K}] \rtimes \mathrm{D}$. For every $g \in \mathrm{G}$, we denote by $\Gamma(g)$ the element of $\mathfrak{c}_0$ such that $g = k' \exp(\Gamma(g))s$.
We denote by Π_j^* the representation of $\mathrm{G}^* = \mathrm{G} \,/\, \mathrm{D} \cap \mathrm{Z(G)} \cap \mathrm{D}$ given by:

$$\Pi_j^*(g^*) = e^{-\mu(\Gamma(g))}\Pi_j(g) \qquad (1 \leq j \leq s),$$

where μ is the form on $\mathfrak{c}_0$ such that $\eta_1(\exp(\Gamma_i)) = e^{\mu(\Gamma_i)}, 1 \leq i \leq r$ (see the proof of Proposition 7) where Γ_i is a basis of $\mathfrak{c}_0$ such that $\exp(\Gamma_i)$ generated $\mathrm{Z(G)} \cap \mathrm{D}$.
We denote by $E^i_{\mathrm{D}} : \mathscr{H}_i \to \mathscr{H}_i^{\mathrm{D}}$ the projection onto the D-isotypic component in $\mathscr{H}_i$. As in Equation (3.5), we get:

$$E^i_{\mathrm{D}} = d(\mathrm{D}) \int_{\mathrm{K}^*} \overline{\chi_{\mathrm{D}^*}(u^*)} \Pi_i^*(u^*) du * .$$

We denote by K_0 the subset of K given by

$$\mathrm{K}_0 = \left\{ k' \exp(\Gamma), \Gamma = \sum_{i=1}^r t_i \Gamma_i, |t_i| \leq \frac{1}{2} \right\} \subseteq \mathrm{K}.$$

Obviously, K_0 is compact. Moreover, for every $k* \in \mathrm{K}^*$, there exists an element $k_0 \in \mathrm{K}_0$ such that $k* = k_0^*$. Indeed, let $k_1 \in K$ such that $k* = k_1^*$. Using the decomposition of $\mathrm{K} = [\mathrm{K}, \mathrm{K}] \rtimes \mathrm{D}$. we get:

$$k_1 = k' \exp(\sum_{i=1}^r t_i \exp(\Gamma_i)) \qquad (t_i \in \mathbb{R}).$$

By definition of K^*, we get that for every $n_1, \ldots, n_r \in \mathbb{Z}$,

$$k* = \left(k_1 \exp(\sum_{i=1}^r n_i \exp(\Gamma_i)) \right)^* = \left(k' \exp(\sum_{i=1}^r (t_i + n_i) \exp(\Gamma_i)) \right)^* .$$

Obviously, we can choose integers $n_1, \ldots, n_r \in \mathbb{Z}$ such that $|t_i + n_i| \leq \frac{1}{2}, 1 \leq i \leq r$.
In particular, it follows easily that

$$E^i_{\mathrm{D}} = d(\mathrm{D}) \int_{\mathrm{K}_0} \overline{\chi_{\mathrm{D}}(u)} \Pi_i(u) du \qquad (1 \leq i \leq s),$$

where the measure on K is chosen such that $du(\mathrm{K}_0) = 1$. We denote by ϕ the function on G given by:

$$\phi = \sum_{i=1}^s c_i \phi^{\mathrm{D}}_{\Pi_i}.$$

Because $c_1 \neq 0$, by Theorem 4.5.3, we get that $\phi \neq 0$. By continuity of ϕ, we can find a function $f \in \mathscr{C}_c^\infty(\mathrm{G})$ such that:

$$\int_{\mathrm{G}} f(g) \phi(g) dg \neq 0.$$

We get:

$$
\begin{aligned}
0 &\neq \int_{\mathrm{G}} f(g)\phi(g)dg = \sum_{i=1}^{s} c_i \int_{\mathrm{G}} f(g)\phi_{\Pi}^{\mathrm{D}}(g)dg \\
&= \sum_{i=1}^{s} c_i \int_{\mathrm{G}} f(g)\,\mathrm{tr}(E_{\mathrm{D}}^{i}\Pi_i(g)E_{\mathrm{D}}^{i})dg = \sum_{i=1}^{s} c_i\,\mathrm{tr} \int_{\mathrm{G}} f(g)E_{\mathrm{D}}^{i}\Pi_i(g)dg \\
&= d(\mathrm{D})\sum_{i=1}^{s} c_i\,\mathrm{tr} \int_{\mathrm{G}} f(g) \int_{\mathrm{K}_0} \overline{\chi_{\mathrm{D}}(u)}\Pi_i(ug)dudg = d(\mathrm{D})\sum_{i=1}^{s} c_i \int_{\mathrm{K}_0}\int_{\mathrm{G}} f(u^{-1}g)\overline{\chi_{\mathrm{D}}(u)}\Pi_i(g)dgdu \\
&= d(\mathrm{D})\sum_{i=1}^{s} c_i \int_{\mathrm{G}} \left(\int \mathrm{K}_0 f(u^{-1}g)\overline{\chi_{\mathrm{D}}(u)}du\right)\Pi_i(g)dg = d(\mathrm{D})\sum_{i=1}^{s} c_i \int_{\mathrm{G}} f_{\mathrm{D}}(g)\Pi_i(g)dg \\
&= \sum_{i=1}^{s} c_i\Theta_{\Pi_i}(f_{\mathrm{D}})
\end{aligned}
$$

where $f_{\mathrm{D}}(g) = \int \mathrm{K}_0 f(u^{-1}g)\overline{\chi_{\mathrm{D}}(u)}du$ is in $\mathscr{C}_c^{\infty}(\mathrm{G})$. Then,

$$
\sum_{i=1}^{s} c_i\Theta_{\Pi_i}(f_{\mathrm{D}}) \neq 0, \tag{4.35}
$$

which is in contradiction with Equation (4.34). In particular, $c_1 = 0$ and the theorem follows.

□

We get the following Corollary.

Corollary 4.5.6. *If* $(\Pi_1, \mathscr{H}_1)$ *and* $(\Pi_2, \mathscr{H}_2)$ *are two quasi-simple irreducible representations of* G. *Then,* $\Theta_{\Pi_1} = \Theta_{\Pi_2}$ *if and only if* Π_1 *and* Π_2 *are infinitesimally equivalent.*

Proof. Direct consequence of Corollary 4.5.4 and Theorem 4.5.5.

□

To finish this chapter, we prove that infinitesimally equivalent can be replaced by equivalent in Corollary 4.5.6 if both representations are unitary.

Theorem 4.5.7. *Let* $(\Pi_1, \mathscr{H}_1)$ *and* $(\Pi_2, \mathscr{H}_2)$ *be two irreducible unitary representations of* G. *Then, the representations* Π_1 *and* Π_2 *are infinitesimally equivalent if and only if they are equivalent.*

Proof. □

Corollary 4.5.8. *Let* $(\Pi_1, \mathscr{H}_1)$ *and* $(\Pi_2, \mathscr{H}_2)$ *be two irreducible unitary representations of* G. *Then, the representations* Π_1 *and* Π_2 *are equivalent if and only if* $\Theta_{\Pi_1} = \Theta_{\Pi_2}$.

Exercice 13. Assume that Π_1 and Π_2 are two infinitesimally equivalent irreducible unitary representations. Using Corollary 3.5.5, we get $\mathscr{H}_1^0 = \sum_{D\in\Omega(K)} \mathscr{H}_D^1$ and $\mathscr{H}_1^0$ is dense in $\mathscr{H}_1$. Prove that

$$\alpha(\mathscr{H}_1^D) = \mathscr{H}_D^2 \qquad (b \in \mathscr{U}(\mathfrak{g}), \Psi \in \mathscr{H}_0^1, D \in \Omega(K)),$$

where α is the map introduced in Definition 4.5.1.

Proof of Theorem 4.5.7. We already know that Π_1 and Π_2 equivalent imply that Π_1 and Π_2 are infinitesimally equivalent. We prove the converse.
The only thing we know about α is that α is $1-1$. We are going to prove that α is continuous and can be extended to a map $\alpha : \mathscr{H}_1^D \to \mathscr{H}_2^D$ (still bijective) such that $\alpha(\Pi_1(g)\Psi) = \Pi_2(g)\alpha(\Psi), \Psi \in \mathscr{H}_1, g \in G$.
The K-representation $\mathscr{H}_1^D$ and $\mathscr{H}_2^D$, $D \in \Omega(K)$, are unitary (and finite-dimensional). According to the Exercice 13, there exists $\beta_D : \mathscr{H}_1^D \to \mathscr{H}_2^D$ an element $U(\mathscr{H}_1^D, \mathscr{H}_2^D)$ such that:

$$\langle \Pi_2(k)\beta_D\Psi_1, \beta_D\Psi_2\rangle_2 = \langle \Pi_1(k)\Psi_1, \Psi_2\rangle_1$$

for every $\Psi_1, \Psi_2 \in \mathscr{H}_1^D, k \in K$ (and β_D satisfies $\beta_D \circ \Pi_1(k) = \Pi_2(k) \circ \beta_D$ for $k \in K$).
Because we can construct such a map for all $D \in \Omega(K)$, we can define a map $\beta : \mathscr{H}_1^0 \to \mathscr{H}_2^0$ such that $\beta_{|_{\mathscr{H}_D^1}} = \beta_D$. The map β is irreducible ($\beta \in U(\mathscr{H}_1^0, \mathscr{H}_2^0)$) and then, the map $S : \mathscr{H}_1^0 \to \mathscr{H}_2^0$ given by:

$$S = \beta^{-1}\alpha$$

is well-defined and bijective. Obviously, because

$$\alpha : \mathscr{H}_1^D \to \mathscr{H}_2^D, \qquad \beta^{-1} : \mathscr{H}_2^D \to \mathscr{H}_1^D, \qquad D \in \Omega(K),$$

then $S(\alpha : \mathscr{H}_1^D) = \mathscr{H}_1^D$. Because $\dim_{\mathscr{C}} \mathscr{H}_1^D < \infty$, there exists $S_D^* : \mathscr{H}_1^D \to \mathscr{H}_1^D$ such that:

$$\langle S\phi, \Psi\rangle = \langle \phi, S^*\Psi\rangle \qquad (\phi, \Psi \in \mathscr{H}_1^D).$$

Then, by orthogonality of the $\mathscr{H}_1^{D}{}'s$, we get a map $S^* : \mathscr{H}_1^0 \to \mathscr{H}_1^0$ such that

$$\langle S\phi, \Psi\rangle = \langle \phi, S^*\Psi\rangle \qquad (\phi, \Psi \in \mathscr{H}_1^0).$$

We denote by A the following invertible operator on $\mathscr{H}_1^0$

$$A = S^*S.$$

We notice that for $X \in \mathfrak{g}_0$ and $\phi, \Psi \in \mathscr{H}_1^0$, we have $\langle \phi, \Pi_1(X)\Psi \rangle = -\langle \Pi_1(X)\phi, \Psi \rangle$. Indeed,

$$\begin{aligned}
\langle \phi, \Pi_1(X)\Psi \rangle &= \lim_{t\to 0} \left\langle \phi, \frac{\Pi_1(\exp(tX))\Psi - \Psi}{t} \right\rangle = \lim_{t\to 0} \frac{1}{t} \left(\left\langle \Pi_1(\exp(tX))^{-1}\phi, \Psi \right\rangle - \langle \phi, \Psi \rangle \right) \\
&= \lim_{t\to 0} \left\langle \frac{\Pi_1(\exp(-tX))\phi - \phi}{t}, \Psi \right\rangle = -\langle \Pi_1(X)\phi, \Psi \rangle
\end{aligned}$$

For every $\phi, \Psi \in \mathscr{H}_1^0$ and $b \in \mathscr{U}(\mathfrak{g})$, we get:

$$\begin{aligned}
\left\langle \phi, A\Pi_1(b)A^{-1}\Psi \right\rangle_1 &= \left\langle \phi, \mathrm{S}^* \mathrm{S}\, \Pi_1(b)\, \mathrm{S}^{-1}\, \mathrm{S}^{*-1}\Psi \right\rangle_1 = \left\langle \mathrm{S}\,\phi, \mathrm{S}\, \Pi_1(b)\, \mathrm{S}^{-1}\, \mathrm{S}^{*-1}\Psi \right\rangle_1 \\
&= \left\langle \beta\, \mathrm{S}\,\phi, \beta\, \mathrm{S}\, \Pi_1(b)\, \mathrm{S}^{-1}\, \mathrm{S}^{*-1}\Psi \right\rangle_2 .
\end{aligned}$$

because $\beta \in \mathrm{U}(\mathscr{H}_1^0, \mathscr{H}_2^0)$. Moreover,

$$\beta\, \mathrm{S}\, \Pi_1(b)\, \mathrm{S}^{-1} = \beta(\beta^{-1}\alpha)\Pi_1(b)\alpha^{-1}\beta = \alpha\Pi_1(b)\alpha^{-1}\beta = \Pi_2(b)\beta.$$

by definition of α. Then,

$$\begin{aligned}
\left\langle \phi, A\Pi_1(b)A^{-1}\Psi \right\rangle_1 &= \left\langle \beta\, \mathrm{S}\,\phi, \beta\, \mathrm{S}\, \Pi_1(b)\, \mathrm{S}^{-1}\, \mathrm{S}^{*-1}\Psi \right\rangle_2 = \left\langle \alpha\phi, \Pi_2(b)\beta \mathrm{S}^{*-1}\Psi \right\rangle_2 \\
&= -\left\langle \Pi_2(b)\alpha\phi, \beta \mathrm{S}^{*-1}\Psi \right\rangle_2 = -\left\langle \alpha\Pi_1(b)\phi, \beta \mathrm{S}^{*-1}\Psi \right\rangle_2 \\
&= -\left\langle \beta^{-1}\alpha\Pi_1(b)\phi, \mathrm{S}^{*-1}\Psi \right\rangle_1 = -\left\langle \mathrm{S}\, \Pi_1(b)\phi, \mathrm{S}^{*-1}\Psi \right\rangle_1 \\
&= -\langle \Pi_1(b)\phi, \Psi \rangle_1 = \langle \phi, \Pi_1(b)\Psi \rangle_1
\end{aligned}$$

In particular, for every $\phi \in \mathscr{H}_1^0$,

$$\left\langle \phi, (A\Pi_1(b)A^{-1} - \Pi_1(b))\Psi \right\rangle_1 = 0.$$

By continuity, we get $\left\langle \phi, (A\Pi_1(b)A^{-1} - \Pi_1(b))\Psi \right\rangle_1 = 0$ for every $\phi \in \mathscr{H}_1$, so

$$A\Pi_1(b)A^{-1}\Psi = \Pi_1(b)\Psi \qquad (\Psi \in \mathscr{H}_1^0). \tag{4.36}$$

Let $\mathrm{D}_0 \in \Omega(\mathrm{K})$. Obviously, the operator $\mathscr{H}_1^{\mathrm{D}_0}$ is A-stable. In particular, according to Schur's Lemma, we can find $\Psi_0 \in \mathscr{H}_1^{\mathrm{D}_0}$ such that $A\Psi_0 = c\Psi_0$ with $c \in \mathbb{C}^8$. We get easily that $c \in \mathbb{R}_+^*$. Indeed,

$$c\, \langle \Psi_0, \Psi_0 \rangle = \langle A\Psi_0, \Psi_0 \rangle = \langle \mathrm{S}^*\, \mathrm{S}\, \Psi_0, \Psi_0 \rangle = \langle \mathrm{S}\, \Psi_0, \mathrm{S}\, \Psi_0 \rangle ,$$

and in particular, $c\|\Psi_0\|_1^2 = \|\,\mathrm{S}\, \Psi_0\|_1^2$, and then $c \in \mathbb{R}_+^*$.

We now prove that $A\Psi = c\Psi$ for every $\Psi \in \mathscr{H}_1^0$. Indeed, because Π_1 is irreducible, $\mathscr{H}_1 = \overline{\Pi_1(\mathscr{U}(\mathfrak{g}))\Psi_0}$ and $\Pi_1(\mathscr{U}(\mathfrak{g})\Psi_0 = \mathscr{H}_1^0$. Then, using Equation 4.36, we get:

$$A\Pi_1(X)\Psi_0 = \Pi_1(X)A\Psi_0 = c\Pi_1(X)\Psi_0 \qquad (X \in \mathscr{U}(\mathfrak{g})).$$

Now, for every $\Psi \in \mathscr{H}_1^0$, we get:

$$\|\alpha\Psi\|_2^2 = \|\beta^{-1}\alpha\Psi\|_1^2 = \| \mathrm{S}\,\Psi\|_1^2 = \langle A\Psi, \Psi\rangle = c\|\Psi\|_1^2.$$

Then, α is continuous and can be extended uniquely to a linear continuous map $\alpha : \mathscr{H}_1 \to \mathscr{H}_2$ satisfying $\|\alpha\Psi\|_2 = \sqrt{c}\|\Psi\|_1$, with $\Psi \in \mathscr{H}_1$.
Moreover, $\alpha(\mathscr{H}_1)$ is closed in $\mathscr{H}_2$ and $\mathscr{H}_2^0 \subseteq \alpha(\mathscr{H}_1)$ because $\mathscr{H}_2^0 = \alpha(\mathscr{H}_1^0) \subseteq \alpha(\mathscr{H}_1)$. Using that $\overline{\mathscr{H}_2^0} = \mathscr{H}_2$, we get:

$$\mathscr{H}_2 = \overline{\mathscr{H}_2^0} \subseteq \overline{\alpha(\mathscr{H}_1)} = \alpha(\mathscr{H}_1) \subseteq \mathscr{H}_2,$$

i.e. $\alpha(\mathscr{H}_1) = \mathscr{H}_2$.
We still have to prove that the map $\alpha : \mathscr{H}_1 \to \mathscr{H}_2$ intertwine Π_1 and Π_2. We let that as an exercice.

Exercice 14. Prove that $\alpha(\Pi_1(g)\Psi) = \Pi_2(g)(\alpha(\Psi)$ for every $g \in \mathrm{G}$ and $\Psi \in \mathscr{H}_1$.

□

Chapter 5

Invariant eigendistributions on semi-simple Lie groups

The goal of this chapter is to prove the following theorem:

Theorem 5.0.1. *Let Ω be a completely invariant open set in* |*G and T be a distribution on Ω. Assume that:*

1. *T is invariant,*
2. *There exists an ideal $\mathscr{U}$ of $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ such that $\dim_{\mathbb{C}}(\mathrm{Z}(\mathscr{U}(\mathfrak{g}))/\mathscr{U}) < \infty$ and $uT = 0$ for every $u \in \mathscr{U}$.*

Then, $T = F_T$, where F_T is in $\mathrm{L}^1_{loc}(\Omega)$ and the restriction of F_T to $\Omega' = \Omega \cap \mathrm{G}'$ is analytic.

In particular, according to Lemma 4.3.2 and Definition 3.5.1, it follows that the character Θ_Π of an irreducible quasi-simple representation $(\Pi, \mathscr{H})$ of G is given by a function $F_\Pi \in \mathrm{L}^1_{loc}(\mathrm{G})$ with is analytic on G'.

5.1 Basics about Lie groups and differential operators

5.1.1 Differential operators on manifolds

Let M be a real connected manifold of dimension n. We denote by $\mathscr{C}^\infty(M)$ the space of smooth functions and $\mathscr{C}^\infty(M)$ the space of compactly supported function on $\mathscr{C}^\infty(M)$.
We denote by $\mathscr{X}(M)$ the set of. derivations of $\mathscr{C}^\infty(M)$, i.e.

$$\mathscr{X}(M) = \{X : \mathscr{C}^\infty(M) \to \mathscr{C}^\infty(M), X(fg) = X(f)g + fX(g)\}.$$

It's the set of $\mathscr{C}^\infty$-vectors fields of M.

Definition 5.1.1. A continuous endomorphism D of $\mathscr{C}_c^\infty(M)$ is called a differential. operator if whenever U is an open set in M and f a function of $\mathscr{C}_c^\infty(M)$. vanishing on U, then Df vanishes on U.

The next proposition will be useful for us.

Proposition 5.1.2. *Let D be a differential operator on M.For each $p \in M$ and each open connected neighbourhood U of p on which the local coordinates system $\Psi : x \to (x_1, \ldots, x_n)$ is valid, there exists a finite set of functions a_α of class $\mathscr{C}^\infty$ such that for each $f \in \mathscr{C}_c^\infty(M)$ with support contained in U,*

$$Df(x) = \begin{cases} \sum\limits_{\alpha=(\alpha_1,\ldots,\alpha_n)} a_\alpha(x) D^\alpha f \circ \Psi^{-1}(x) & x \in U \\ 0 & \textit{otherwise} \end{cases}$$

Proof. The proof of this result can be found in [14, Proposition 1].

□

Notation 5.1.3. We denote by $\mathrm{D}(M)$ the set of differential operators.

From now on, we assume that $\mathrm{G} = \mathrm{M}$ is a connected Lie group. We denote by $(\mathrm{L}, \mathrm{L}^2(\mathrm{G}, dg))$ the left regular representation. Obviously, the space $\mathscr{C}_c^\infty(\mathrm{G})$ is G-invariant.
We define an action of G on $\mathrm{D}(\mathrm{G})$ by

$$(\tau(g)D)(f) = L_g \circ D(f \circ L_{g^{-1}}) \qquad (g \in \mathrm{G}, f \in \mathscr{C}_c^\infty(\mathrm{G}), D \in \mathrm{D}(\mathrm{G})).$$

Definition 5.1.4. We say that $\mathrm{D} \in \mathrm{D}(\mathrm{G})$ is left-invariant if $\tau(g)D = D$ for all $g \in \mathrm{G}$, i.e. $L_g \circ D(f) = D(f \circ L_g)$.

We denote by $\mathrm{D}_\mathrm{G}(\mathrm{G})$ the set of left-invariant differential operators of G.
We say that D is right invariant if $\tau_1(g)(D) = D$ for every $g \in \mathrm{G}$, where

$$(\tau_1(g)D)(f) = R_{g^{-1}} \circ D(f \circ R_g) \qquad (f \in \mathscr{C}_c^\infty(\mathrm{G}).$$

The operator D is is said to be invariant if $\tau(g)\tau_1(g)(D) = D$ for every $g \in \mathrm{G}$, i.e. $R_{g^{-1}} \circ L_g \circ D(L_{g^{-q}} \circ f \circ R_g) = D(f)$ for every $f \in \mathscr{C}_c^\infty(\Omega)$.

Notation 5.1.5. We denote by $\mathrm{D}^\mathrm{G}(\mathrm{G})$ the set of right-invariant differential operators and by $\mathrm{D}_\mathrm{G}^\mathrm{G}(\mathrm{G})$ the set of bi-invariant differential operators.

We can also define this action on the set of vector fields $\mathscr{X}(\mathrm{G})$. We denote by $\mathscr{X}_\mathrm{G}(\mathrm{G})$ the set of left-invariant vector fields of $\mathscr{X}(\mathrm{G})$. We know that

$$\mathscr{X}_\mathrm{G}(\mathrm{G}) \ni X \to X_e \in \mathfrak{g}, \tag{5.1}$$

is an isomorphism of Lie algebras, where $X_e(f) = (Xf)(e)$ We recall the following result.

Theorem 5.1.6. *The natural embedding*

$$\mathfrak{g}(\approx \mathscr{X}_{\mathrm{G}}(\mathrm{G})) \to \mathrm{D}_{\mathrm{G}}(\mathrm{G})$$

extends to an algebra isomorphism

$$\mathscr{U}(\mathfrak{g}) \to \mathrm{D}_{\mathrm{G}}(\mathrm{G}). \tag{5.2}$$

Proof. The proof of this result can be found in [15].

□

We now define the adjoint of an operator.

Lemma 5.1.7. *Let $D \in \mathrm{D}(\mathrm{G})$. There exists a unique differential operator $D^* \in \mathrm{D}(\mathrm{G})$ such that for every $f_1, f_2 \in \mathscr{C}_c^\infty(\mathrm{G})$, we get:*

$$\int_{\mathrm{G}} Df_1(g)f_2(g)dg = \int_{\mathrm{G}} f_1(g)D^* f_2(g)dg$$

Moreover, if $D \in \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G})$ if and only if $D^* \in \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G})$.

Proof. The existence of D^* is well-known. Moreover, for every $f_1, f_2 \in \mathscr{C}_c^\infty(\Omega)$, we get:

$$\begin{aligned}
&\int_\Omega D^*(f_1 \circ L_{g^{-1}} \circ R_g)(w)f_2(w)dw = \int_\Omega f_1 \circ L_{g^{-1}} \circ R_g(w)D(f_2)(w)dw \\
= &\int_\Omega f_1(g^{-1}wg)Df_2(w)dw = \int_\Omega f_1(w)Df_2(gwg^{-1})dw \\
= &\int_\Omega f_1(w)Df_2 \circ L_g \circ R_{g^{-1}}(w)dw = \int_\Omega f_1(w)D(f_2 \circ L_g \circ R_{g^{-1}})(w)dw \\
= &\int_\Omega D^* f_1(w)f_2 \circ L_g \circ R_{g^{-1}}(w)dw = \int_\Omega D^* f_1(g^{-1}wg)f_2(w)dw \\
= &\int_\Omega (D^* f_1) \circ L_{g^{-1}} \circ R_g(w)f_2(w)dw
\end{aligned}$$

In particular, $D^*(f_1 \circ L_{g^{-1}} \circ R_g) = (D^* f_1) \circ L_{g^{-1}} \circ R_g$ and then $D^* \in \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G})$.

□

Exercice 15. Prove that for every $D_1, D_2 \in \mathrm{D}(\mathrm{G})$, $(D_1D_2)^* = D_2^*D_1^*$.

Let $D \in \mathrm{D}(\mathrm{G})$. Obviously, $D \in \mathrm{End}(\mathscr{C}_c^\infty(\mathrm{G}))$ can be extended naturally to an element of $\mathrm{End}(\mathscr{D}'(\mathrm{G}))$ by:

$$D(T)(f) = T(D^*(f)) \qquad (T \in \mathscr{D}'(\mathrm{G}), f \in \mathscr{C}_c^\infty(\mathrm{G})).$$

Exercice 16. Prove that for $D \in \mathscr{D}'(\mathrm{G})$ and $T \in \mathscr{C}_c^\infty(\mathrm{G})$, $DT \in \mathscr{D}'(\mathrm{G})$. Moreover, prove that $\mathrm{D}(\mathrm{G})$ acts on $\mathscr{D}'(\mathrm{G})$.

Definition 5.1.8.
1. A function $f \in \mathscr{C}_c^\infty(\mathrm{G})$ is invariant if $f^g = f$ for every $g \in \mathrm{G}$, where f^g is the element of $\mathscr{C}_c^\infty(\mathrm{G})$ given by $f^g(y) = f(gyg^{-1}), y \in \mathrm{G}$.
2. A distribution T on G is G-invariant if $T(f^g) = T(f)$ for all $g \in \mathrm{G}, f \in \mathscr{C}_c^\infty(\mathrm{G})$.
3. A distribution T on G is an eigendistribution if $D(T) = \chi_T(D)T$ for all $D \in \mathrm{D}_\mathrm{G}^\mathrm{G}(\mathrm{G})$, where $\chi : \mathrm{D}_\mathrm{G}^\mathrm{G}(\mathrm{G}) \to \mathbb{C}$ is an algebra homomorphism.

Exercice 17. Prove that via the map given in Equation (5.2), $\mathrm{Im}(\mathrm{Z}(\mathscr{U}(\mathfrak{g}))) = \mathrm{D}_\mathrm{G}^\mathrm{G}(\mathrm{G})$. Moreover, for every irreducible quasi-simple representation of G, Θ_Π is an eigendistribution.

We now prove some basic results about differential operators. Let Ω be a G-invariant subspace (i.e. for every $g \in \mathrm{G}$, $g\Omega g^{-1} \subseteq \Omega$ and D be a differential operator on Ω.

Notation 5.1.9. We denote by $\mathscr{I}_\mathrm{G}(\Omega)$ the set of G-invariant functions of $\mathscr{C}_c^\infty(\Omega)$ and by $\widetilde{\mathscr{I}_\mathrm{G}}(\Omega)$ the set of invariant definitions.

Lemma 5.1.10. *Let $D \in \mathrm{D}_\mathrm{G}^\mathrm{G}(\mathrm{G})$ and $T \in \widetilde{\mathscr{I}_\mathrm{G}}(\Omega)$. Then, $DT \in \widetilde{\mathscr{I}_\mathrm{G}}(\Omega)$.*

Proof. Let $g \in \mathrm{G}$. For every $f \in \mathscr{C}_c^\infty(\Omega)$, we get:

$$\begin{aligned} DT(f^g) &= T(D^*(f^g)) = T(D^*(f \circ L_{g^{-1}} \circ R_g)) \\ &= T((D^*f) \circ L_{g^{-1}} \circ R_g) \qquad (\text{ because } D^* \in \mathrm{D}_\mathrm{G}^\mathrm{G}(\mathrm{G})) \\ &= T(D^*f) \qquad (\text{ because } T \in \widetilde{\mathscr{I}_\mathrm{G}}(\Omega)) \\ &= DT(f) \end{aligned}$$

□

We recall the following theorem.

Theorem 5.1.11. *Let $T \in \mathscr{D}'(\Omega)$ such that for every $w \in \Omega$, there exists V_w an open neighbourhood of w in Ω, $T_{|V_w} = 0$. Then, $T = 0$*

It motivates the following definition.

Definition 5.1.12. The support of T is the set of $w \in \Omega$ such that we cannot find an open neighbourhood V_w of w such that $T_{|V_w} = 0$. We will denote by $\mathrm{supp}(T)$ the support of T.

Exercice 18. Prove that $\mathrm{supp}(T)$ is closed in Ω. Moreover, if $T = T_f$ for a function $f \in \mathscr{C}_c^\infty(\Omega)$, then $\mathrm{supp}(T) = \mathrm{supp}(f)$.

Lemma 5.1.13. *Let $T \in \widetilde{\mathscr{I}_{\mathrm{G}}}(\Omega)$. Then,* $\mathrm{supp}(T)$ *is* G*-invariant.*

Proof. Let $g \in \mathrm{G}$ and $x \in \mathrm{supp}(T)$. Assume that $gxg^{-1} \notin \mathrm{supp}(T)$. Then, there exists an open neighbourhood $V_{gxg^{-1}}$ of gxg^{-1} such that $T_{|V_{gxg^{-1}}} = 0$.
We denote by $V_x = L_{g^{-1}} \circ R_g(V_{gxg^{-1}})$. Then, V_x is an open neighbourhood of x. Moreover, for every $f \in \mathscr{C}_c^\infty(\Omega)$ such that $\mathrm{supp}(f) \subseteq V_x$, then, $f^{g^{-1}}$ is in $\mathscr{C}_c^\infty(\Omega)$ and $\mathrm{supp}(f^{g^{-1}}) \subseteq V_{gxg^{-1}}$. In particular, $T(f) = T(f^{g^{-1}}) = 0$ and then $T_{|V_x} = 0$, which is not possible because $x \in \mathrm{supp}(T)$.

□

5.1.2 The set of regular points

Here we assume that G is a semi-simple connected Lie group of dimension n. Let $x \in \mathrm{G}$. We define the function $\mathrm{D}_x : \mathbb{C} \to \mathbb{C}$ by:

$$D_x(t) = \det((t+1) - \mathrm{Ad}(x)).$$

We obviously get that:

$$D_x(t) = \sum_{0 \leq j \leq n} t^j D_j(x).$$

The maps $D_j : \mathrm{G} \to \mathbb{C}$ are analytics and $D_n = 1$. We denote by l the rank of G.

Definition 5.1.14. The set of regular points G' of G is defined by:

$$\mathrm{G}' = \{g \in \mathrm{G}, D_l(g) \neq 0\}.$$

We get the following proposition.

Proposition 5.1.15. *The set* G' *is open and dense in* G*. Moreover,* $dg(\mathrm{G} / \mathrm{G}') = 0$*, where* dg *is a Haar measure of* G

Let's see that on an example.

Example 5.1.16. In this example, we consider $\mathrm{G} = \mathrm{U}(p, q, \mathbb{C})$ and $\mathrm{T} \approx \mathrm{S^1}^{p+q}$. Then,

$$\mathrm{T}' = \left\{\mathrm{diag}(t_1, \ldots, t_{p+q}), t_i \neq t_j, i \neq j\right\}.$$

Indeed, let $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$ and $\xi_\alpha : \mathrm{T} \to \mathrm{S}^1$ the corresponding character of T. For every $X \in \mathfrak{g}_\alpha, \alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$, we have $\mathrm{Ad}(t)X = \xi_\alpha(t)X$, where $t \in \mathrm{T}$. According to Section (CITE), we get $\alpha = e_j - e_k, 1 \leq j < k \leq p+q$, we get:

$$\xi_{e_j - e_k}((t_1, \ldots, t_{p+q}) = \exp(ih_1, \ldots, ih_{p+q})) = e^{(e_j - e_k)(ih_1, \ldots, ih_{p+q})} = e^{ih_j} e^{-ih_k} = t_j t_k^{-1}.$$

Then,

$$\det((u+1)-\mathrm{Ad}(t)) = t^{p+q}\prod_{1\le i<j\le p+q}(u+1-t_it_j^{-1})(u+1-t_i^{-1}t_j).$$

Then,

$$D_{p+q}(t) = \prod_{1\le i<j\le p+q}(1-t_it_j^{-1})(1-t_i^{-1}t_j) = \pm\prod_{1\le i<j\le p+q}t_i^{-1}t_j^{-1}(t_i-t_j).$$

and $D_{p+q}(t)\neq 0$ if and only if $t_i - t_j \neq 0, i\neq j$.
For $g\in \mathrm{G}$ and $t\in \mathrm{T}'$, then, $gtg^{-1}\in \mathrm{G}'$. Indeed,

$$\begin{aligned} D_{gtg^{-1}}(u) &= \det((u+1)\,\mathrm{Id}-\mathrm{Ad}(gtg^{-1})) = \det(\mathrm{Ad}(g)((u+1)\,\mathrm{Id}-\mathrm{Ad}(t))\,\mathrm{Ad}(g^{-1})) \\ &= \det((u+1)\,\mathrm{Id}-\mathrm{Ad}(t)) = D_t(u) \end{aligned}$$

Then, $D_{p+q}(gtg^1) = D_{p+q}(t)$.

As in Section 2.2, an element $x\in \mathrm{G}$ is said to be semi-simple if $\mathrm{Ad}(x)$ is semi-simple.

Notation 5.1.17. For an element $x\in\mathfrak{g}_0$, we denote by $\mathfrak{z}_x$ the centraliser of x in $\mathfrak{g}_0$, i.e.

$$\mathfrak{z}_x = \{y\in\mathfrak{g}_0, [x,y]=0\}.$$

We admit the following lemma (the proof can be found in [12, Lemma 5]).

Lemma 5.1.18. *Let x be an element of* G. *Then, $x\in \mathrm{G}'$ if and only if $\mathfrak{z}_x$ is a Cartan subalgebra of* $\mathfrak{g}_0$. *Moreover, if x is semisimple, then $\mathfrak{z}_x$ is reductive in $\mathfrak{g}_0$ and* $\mathrm{rk}(\mathfrak{z}_x) = \mathrm{rk}(\mathfrak{g}_0)$. *Finally, a regular element is always semi-simple.*

We recall the abstract decomposition of an element $x\in \mathrm{G}$ (consequence of the one given in Section 2.2 for Lie algebras). There exists elements $h,n\in \mathrm{G}$ such that $x = hn$, $\mathrm{Ad}(h)$ (resp. $\mathrm{Ad}(n)$) is semisimple (resp. nilpotent) and $hn = nh$.
The proof of the following is easy and let as an exercise.

Lemma 5.1.19. *If $x = hn$ as before. Then, $h\in\bar{\theta}_x$.*

We now recall the concept of completely invariant subspace of G.

Definition 5.1.20. A subset Ω of G is said completely invariant if for any compact subset C of U, then

$$\overline{\{gCg^{-1}, g\in \mathrm{G}\}}\subset\Omega.$$

We finish this section with an easy but important property of those spaces.

Lemma 5.1.21. *Let Ω be a completely invariant subspace of U and V a closed invariant subset of Ω. If V contains no semisimple element of Ω, V is empty.*

Proof. Let $x \in V$. According to the previous discussion, we get that $x = hn$ with $h \in \overline{\theta_x}$ (Lemma 5.1.19). Because V is G-invariant, $\theta_x \subseteq V$ and using that V is closed, we get $\overline{\theta_x} \subseteq \overline{V} = V$. In particular, $h \in V$.

□

5.1.3 A particular completely invariant subspace

In this section, we assume that the real Lie algebra $\mathfrak{g}_0$ is semi-simple. Let $c > 0$. We denote by $\mathfrak{g}_0(c)$ the set of elements $X \in \mathfrak{g}_0$ such that $|\operatorname{Im}(\lambda)| < c$ for every $\lambda \in \operatorname{Spec}(\operatorname{ad}(X))$. Let $g \in \mathrm{G}$ and $X \in \mathfrak{g}_0(c)$. Then,

$$\operatorname{Spec}(\operatorname{ad}(\operatorname{Ad}(g)X)) = \operatorname{Spec}(\operatorname{ad}(X)).$$

In particular, $\operatorname{Ad}(g)X \in \mathfrak{g}_0(c)$ and then, $\mathfrak{g}_0(c)$ is G-invariant.
Moreover, $\mathfrak{g}_0(c)$ is open in $\mathfrak{g}_0$ and the set of nilpotent elements $\mathscr{N}$ of $\mathfrak{g}_0$ is included in $\mathfrak{g}_0(c)$.

Exercice 19. Prove that $\mathfrak{g}_0(c)$ is a completely invariant subspace of $\mathfrak{g}_0$.

We will admit the following lemma, but explain clearly how it can be applied and used.

Lemma 5.1.22. *Assume that $c \leq \pi$. Then, the exponential mapping from $\mathfrak{g}_0$ into* G *is everywhere regular and injective on $\mathfrak{g}_0(c)$. Moreover, $\Omega(c) = \exp(\mathfrak{g}_0(c))$ is completely invariant in* G.

We now explain how we can transfer differential on $\Omega(c)$ to $\mathfrak{g}_0(c)$.
Fix $0 < c \leq \pi$. For every $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0(c))$, we define a function f_ϕ on $\Omega(c)$ by:

$$f_\phi(\exp(X)) = p(X)^{-1}\phi(X) \qquad (X \in \mathfrak{g}_0(c)), \tag{5.3}$$

where $p(X) = \left|\det\left(\frac{e^{\operatorname{ad}(X/2)} - e^{-\operatorname{ad}(X/2)}}{\operatorname{ad}(X/2)}\right)\right|^{\frac{1}{2}}$. Obviously, f_ϕ is analytic if and only if ϕ is analytic. Moreover,

$$\mathscr{C}_c^\infty(\mathfrak{g}_0(c)) \ni \phi \to f_\phi \in \mathscr{C}_c^\infty(\Omega(c))$$

is a linear topological map. In particular, for any differential operator D on $\Omega(c)$, , there exists a differential operator $\xi(D)$ on $\mathfrak{g}_0(c)$ such that:

$$Df_\phi = f_{\xi(D)\phi} \qquad (\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0(c))). \tag{5.4}$$

Obviously, if D is analytic, $\xi(D)$ is analytic.
We denote by dX the euclidian measure on $\mathfrak{g}_0$ and dg the Haar measure on G. If dX is suitably normalized, we get:

$$dg = p(X)^2 dX \qquad (x = \exp(X), X \in \mathfrak{g}_0(c)).$$

In this case, we get for every $\phi_1, \phi_2 \in \mathscr{C}_c^\infty(\mathfrak{g}_0(c))$ that:

$$\int_{\mathrm{G}} f_{\phi_1}(g) f_{\phi_2}(g) dg = \int_{\mathfrak{g}_0} \phi_1(X)\phi_2(X) dX.$$

Lemma 5.1.23. *For every $D \in \mathrm{D}(\Omega(c))$, $\xi(D)^* = \xi(D^*)$ (where $*$ denote the adjoint of a differential operator, see Lemma 5.1.7).*

Proof. For every $\phi_1, \phi_2 \in \mathscr{C}_c^\infty(\mathfrak{g}_0(c))$, we get:

$$\begin{aligned}
&\int_{\Omega(c)} f_{\xi(D^*)\phi_1}(g) f_{\phi_2}(g) dg = \int_{\Omega(c)} D^* f_{\phi_1}(g) f_{\phi_2}(g) dg \\
= &\int_{\Omega(c)} f_{\phi_1}(g) D f_{\phi_2}(g) dg = \int_{\Omega(c)} f_{\phi_1}(g) f_{\xi(D)\phi_2}(g) dg \\
= &\int_{\mathfrak{g}_0(c)} \phi_1(X)\xi(D)\phi_2(X) dX = \int_{\mathfrak{g}_0(c)} \xi(D)^*\phi_1(X)\phi_2(X) dX \\
= &\int_{\mathfrak{g}_0(c)} f_{\xi(D)^*\phi_1}(X)\phi_2(X) dX
\end{aligned}$$

In particular,

$$\int_{\Omega(c)} \left(f_{\xi(D^*)\phi_1} - f_{\xi(D)^*\phi_1}\right)(g)\phi_2(g) dg = 0,$$

and it implies that $f_{\xi(D^*)\phi_1} = f_{\xi(D)^*\phi_1}$. In particular, $\xi(D^*) = \xi(D)^*$.

□

Similarly, every distribution T on $\Omega(c)$ defines a distribution η_T on $\mathfrak{g}_0(c)$ by:

$$\lambda_T(\phi) = T(f_\phi) \qquad (\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0(c))). \tag{5.5}$$

Lemma 5.1.24. *For every $T \in \mathrm{D}'(\Omega(c))$ and $D \in \mathrm{D}(\Omega(c))$, $\lambda_{DT} = \xi(D)\eta_T$.*

Proof. For every $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0(c))$

$$\begin{aligned}
\lambda_{DT}(\phi) &= DT(f_\phi) = T(D^*(f_\phi)) = T(f_{\xi(D^*)\phi}) \\
&= T(f_{\xi(D)^*\phi}) = T(\xi(D)^*(f_\phi)) \\
&= \xi(D)T(f_\phi) = \xi(D)\lambda_T(\phi)
\end{aligned}$$

□

Exercice 20. Prove that the differential operator $\xi(D)$ if G-invariant for every $D \in \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\Omega(c))$. Similarly, prove that η_T is G-invariant if T is invariant.

Remark 5.1.25. Obviously, all the previous results can be generalised and we can replace $\mathfrak{g}_0(c)$ by a completely invariant subset U of $\mathfrak{g}_0$ such that the restriction of the exponential map to U is injective.

5.2 The map Γ

For $X \in \mathfrak{g}_0$, we denote by L_X and R_X the left and right translation on $\mathscr{U}(\mathfrak{g})$, i.e.

$$L_X : \mathscr{U}(\mathfrak{g}) \ni Y \to L_X(Y) = XY \in \mathscr{U}(\mathfrak{g}).$$

For $x \in \mathrm{G}$, we define the map $\sigma_x : \mathfrak{g} \to \mathrm{End}(\mathscr{U}(\mathfrak{g}))$ given by

$$\sigma_x(X) = L_{\mathrm{Ad}(x^{-1})X} - R_X.$$

Lemma 5.2.1. *For every $X, Y \in \mathfrak{g}_0$, we get* $[\sigma_x(X), \sigma_x(Y)] = \sigma_x([X, Y])$.

Proof. For every element $Z \in \mathscr{U}(\mathfrak{g})$, we get:

$$\begin{aligned}
[\sigma_x(X), \sigma_x(Y)] &= \sigma_x(X)\sigma_x(Y) - \sigma_x(Y)\sigma_x(X) \\
&= \sigma_x(X)(\mathrm{Ad}(x^{-1})(Y)Z - ZY) - \sigma_x(Y)(Ad(x^{-1})(X)Z - ZX) \\
&= \mathrm{Ad}(x^{-1})(X)(\mathrm{Ad}(x^{-1})(Y)Z - ZY) - (\mathrm{Ad}(x^{-1})(Y)Z - ZY)X \\
&- \mathrm{Ad}(x^{-1})(Y)(Ad(x^{-1})(X)Z - ZX) + (Ad(x^{-1})(X)Z - ZX)Y \\
&= (\mathrm{Ad}(x^{-1})(X)\,\mathrm{Ad}(x^{-1})(Y) - \mathrm{Ad}(x^{-1})(Y)\,\mathrm{Ad}(x^{-1})(X))(Z) + ZYX - ZXY \\
&= [\mathrm{Ad}(x^{-1})X, \mathrm{Ad}(x^{-1})Y]Z - Z[X, Y] = (L_{[\mathrm{Ad}(x^{-1})X, \mathrm{Ad}(x^{-1})Y]} - R_{[X,Y]})(Z) \\
&= L_{\mathrm{Ad}(x^{-1})[X,Y]} - R_{[X,Y]})(Z) = \sigma_x([X, Y])
\end{aligned}$$

□

The map σ_x is a representation of $\mathfrak{g}_0$ on $\mathscr{U}(\mathfrak{g})$ and can be extended to $\mathscr{U}(\mathfrak{g})$. We denote by Γ_x the map

$$\Gamma_x : \mathscr{U}(\mathfrak{g}) \otimes \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$$

given by

$$\Gamma_x(X_1 \otimes X_2) = \sigma_x(X_1 X_2.$$

Remark 5.2.2. We denote by $\mathrm{S}(\mathfrak{g})$ the symmetric algebra of $\mathfrak{g}$, i.e.

$$\mathrm{S}(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} \mathfrak{g}^{\otimes k} / \langle X \otimes Y - Y \otimes X, X, Y \in \mathfrak{g} \rangle .$$

Let $\{X_1, \ldots, X_n\}$ be a basis of $\mathfrak{g}$ over $\mathbb{C}$. Then, the elements $X_1^{e_1} \ldots X_n^{e_n}, e_1, \ldots, e_n \in \mathbb{Z}_+$ form a basis of $\mathrm{S}(\mathfrak{g})$. We define by $\lambda : \mathrm{S}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{g})$ the linear map given by:

$$\lambda(X_1^{e_1} \ldots X_n^{e_n}) = \frac{1}{m!} \sum_{(i_1,\ldots,i_m)\in\Omega(e_1,\ldots,e_n)} X_{i_1} \ldots X_{i_m}$$

where $m = \sum_{i=1}^{n} e_i$ and $\Omega(e_1, \ldots, e_n) = \left\{(i_1, \ldots, i_m) \in \mathbb{Z}_+^m, \sharp\{i_u, , i_u = j\} = e_j, 1 \leq j \leq m\right\}$.
One can prove that the map λ is $1-1$. The $\mathbb{Z}$-gradation of the tensor algebra $\mathrm{T}(\mathfrak{g})$ gives a natural gradation on $\mathrm{S}(\mathfrak{g})$

$$\mathrm{S}(\mathfrak{g}) = \bigoplus_{d=0}^{\infty} \mathrm{S}^d(\mathfrak{g}).$$

Notation 5.2.3. We denote by $\mathscr{U}^d(\mathfrak{g}) = \lambda(\mathrm{S}^d(\mathfrak{g})$ and by $\mathscr{U}_d(\mathfrak{g})$ the following space:

$$\mathscr{U}_d(\mathfrak{g}) = \sum_{0 \leq m \leq d} \mathscr{U}^m(\mathfrak{g}).$$

We get directly that for every $x \in \mathrm{G}$,

$$\Gamma_x(\mathscr{U}^{d_1}(\mathfrak{g}) \otimes \mathscr{U}^{d_2}(\mathfrak{g}) \subseteq \mathscr{U}_{d_1+d_2}(\mathfrak{g}).$$

Lemma 5.2.4. *1. Let $\beta : \mathrm{G} \to \mathrm{G}$ be an automorphism of G. For any $x \in \mathrm{G}$ and $X_1, X_2 \in \mathscr{U}(\mathfrak{g})$, we get:*

$$\Gamma_{\beta(x)}(\beta(X_1) \otimes \beta(X_2)) = \beta(\Gamma_x(X_1 \otimes X_2)).$$

2. *Let $X_1, \ldots, X_d$ be elements of $\mathfrak{g}$. Then, for every $\sigma \in \mathscr{S}_d$, the element*

$$X_1 \ldots X_d - X_{\sigma(1)} \ldots X_{\sigma(d)} \in \mathscr{U}_{d-1}(\mathfrak{g}).$$

3. *Let $X_1, \ldots, X_r, Y_1, \ldots, Y_s$ be elements of $\mathfrak{g}$ and $x \in \mathrm{G}$. For all $1 \leq i \leq r$, we denote by $X_i' = \mathrm{Ad}(x^{-1})X_i - X_i = \sigma_x(X_i)$. Then,*

$$\Gamma_x(\lambda(X_1 \ldots X_r) \otimes \lambda(Y_1 \ldots Y_s)) = \lambda(X_1' \ldots X_r' Y_1 \ldots Y_s) \mod \mathscr{U}_{r+s-1}(\mathfrak{g}).$$

Remark 5.2.5 (Remark before proof). We still denote by $\beta : \mathfrak{g}_0 \to \mathfrak{g}_0$ the differential of β. In particular, for every $X, Y \in \mathfrak{g}_0$, $[\beta(X), \beta(Y)] = \beta([X, Y])$. This map is still bijective. Indeed, let $X \in$ such that $\beta(X) = 0$. In particular, for every $t \in \mathbb{R}$, $\beta(tX) = 0$. Then, $\beta(\exp(tX)) = \exp\beta(tX) = e$. Using that $\beta(e) = e$, we get that $\exp(tX) = \exp(0)$. For t small enough, we get $X = 0$.

Proof. 1. For $X, Y \in \mathfrak{g}_0$, we get:

$$\begin{aligned}
& (\beta \circ \sigma_x(X) \circ \beta^{-1})(Y) = \beta \circ \sigma_x(X)(\beta^{-1}(Y)) \\
= \;& \beta(\sigma_x(X)\beta^{-1}(Y)) = \beta(\mathrm{Ad}(x^{-1})X\beta^{-1}(Y) - \beta^{-1}(Y)X) \\
= \;& \beta(\mathrm{Ad}(x^{-1})X)Y - Y\beta(X) = (L_{\beta(\mathrm{Ad}(x^{-1})X)} - R_{\beta(X)})(Y) \\
= \;& (L_{\mathrm{Ad}(\beta(x^{-1})} - R_{\beta(X)})(Y) = \sigma_{\beta(x)}\beta(X)(Y)
\end{aligned}$$

Then, for $X, Y \in \mathscr{U}(\mathfrak{g})$, we get:

$$\begin{aligned}\Gamma_{\beta(x)}(\beta(X)\otimes\beta(Y)) &= \sigma_{\beta(x)}(\beta(X))\beta(Y) = \beta\circ\sigma_x(X)\circ\beta^{-1}(\beta(Y)) \\ &= \beta(\sigma_x Y) = \beta(\Gamma_x(X\otimes Y)\end{aligned}$$

2. It's a well-known fact. Let's see that on an example instead. Let $X_1, X_2, X_3 \in \mathfrak{g}$. Then,

$$\begin{aligned}X_1X_2X_3 - X_2X_3X_1 &= (X_2X_1 + [X_1,X_2].1)X_3 - X_2X_3X_1 = X_2(X_1X_3 - X_3X_1) + [X_1,X_2]X_3 \\ &= X_2[X_1,X_3] + [X_1,X_2]X_3 \in \mathscr{U}_2(\mathfrak{g}).\end{aligned}$$

3. We prove that by induction on r. For $r = 1$,

$$\begin{aligned}\Gamma_x(\lambda(X_1)\otimes\lambda(Y_1\dots Y_s)) &= \mathrm{Ad}(x^{-1})X_1\lambda(Y_1\dots Y_s) - \lambda(Y_1\dots Y_s)X_1 \\ &= \mathrm{Ad}(x^{-1})X_1\lambda(Y_1\dots Y_s) - X_1\lambda(Y_1\dots Y_s) + \mathscr{U}_s(\mathfrak{g}) \\ &= X_1'\lambda(Y_1\dots Y_s) + \mathscr{U}_s(\mathfrak{g}).\end{aligned}$$

The end of the proof is let to the reader.

□

5.3 The map α

In this section, a will denote a semisimple point in G, $\mathfrak{z}_a$ will be the centralizer of $a \in \mathfrak{g}_0$ and Θ_a the analytic subgroup of G such that $\mathrm{Lie}(\Theta(a)) = \mathfrak{z}(a)$.
On the space $\Theta(a)$, we define a function ν_a by:

$$\nu_a(y) = \det(\mathrm{Ad}(ay)^{-1} - \mathrm{Id})_{\mathfrak{g}_0/\mathfrak{z}(a)}.$$

We denote by $\Theta'(a) = \{y \in \Theta(a), \nu_a(y) \neq 0\}$. This space is an open neighbourhood of 1 in $\Theta(a)$ (obviously, $\nu_a(1) \neq 0$). Moreover, ν_a is analytic.
According to Section 3.1, $\mathfrak{z}_a$ is reductive. There exists a subspace $\mathfrak{q}_a$ of $\mathfrak{g}_0 = \mathfrak{z}_a \oplus \mathfrak{q}_a$, where $[\mathfrak{z}_a, \mathfrak{q}_a] \subseteq \mathfrak{q}_a$. Then,

$$\mathfrak{g} = \mathfrak{z}_a^{\mathbb{C}} \oplus \mathfrak{q}_a^{\mathbb{C}}, \qquad [\mathfrak{z}_a^{\mathbb{C}}, \mathfrak{q}_a^{\mathbb{C}}] \subseteq \mathfrak{q}_a^{\mathbb{C}}.$$

Remark 5.3.1. The space $\mathfrak{q}_a^{\mathbb{C}}$ is not a subalgebra of $\mathfrak{g}$. So it does not make sense to consider $\mathscr{U}(\mathfrak{q}_a^{\mathbb{C}})$. But, because $\mathfrak{q}_a^{\mathbb{C}}$ is a subspace of $\mathfrak{g}$, $S(\mathfrak{q}_a^{\mathbb{C}})$ is a subalgebra of $S(\mathfrak{g})$. We denote by $\mathscr{U}(\mathfrak{q}_a^{\mathbb{C}}) = \lambda(S(\mathscr{U}(\mathfrak{q}_a^{\mathbb{C}})))$.

We get the following lemma.

Lemma 5.3.2. *Let $y \in \Theta'(a)$. Then, the map Γ_{ay} restricted to $\mathscr{U}(\mathfrak{q}_a^{\mathbb{C}}) \otimes \mathscr{U}(\mathfrak{z}_a^{\mathbb{C}})$ is bijective.*

Moreover,

$$\sum_{d_1+d_2\le d} \Gamma_{ay}(\mathscr{U}^{d_1}(\mathfrak{q}_a^{\mathbb{C}}) \otimes \mathscr{U}^{d_2}(\mathfrak{z}^{\mathbb{C}})_a) = \mathscr{U}_d(\mathfrak{g}),$$

where $\mathscr{U}^{d_1}(\mathfrak{q}_a^{\mathbb{C}}) = \lambda(\mathrm{S}^{d_1}(\mathfrak{q}_a^{\mathbb{C}}))$.

Notation 5.3.3. We denote by $\mathscr{U}_+(\mathfrak{q}_a^{\mathbb{C}}) = \lambda\left(\sum_{d\ge 1} \mathrm{S}^d(\mathfrak{q}_a^{\mathbb{C}})\right)$.

Corollary 5.3.4. *Let* $X \in \mathscr{U}(\mathfrak{g})$*. Then, for any* $y \in \Theta'(a)$*, there exists unique elements* $\alpha_y(X) \in \mathbb{C} \otimes \mathscr{U}(\mathfrak{z}_a^{\mathbb{C}}) = \mathscr{U}(\mathfrak{z}_a^{\mathbb{C}})$ *and* $\beta_y(X) \in \mathscr{U}_+(\mathfrak{q}_a^{\mathbb{C}}) \otimes \mathscr{U}(\mathfrak{z}_a^{\mathbb{C}})$ *such that:*

$$X = \alpha_y(X) + \Gamma_{ay}(\beta_y(X)).$$

Proof. Obvious using Lemma 5.4.1 and that for every $Z \in \mathbb{C} \otimes \mathscr{U}(\mathfrak{z}_a^{\mathbb{C}})$, $\Gamma_{ay}(Z) = Z$.

□

Corollary 5.3.5. *Let* $\mathrm{Z}(a)$ *be the centraliser of* $a \in \mathrm{G}$*. Then, if* $x \in \mathrm{Z}(a)$*,* $y \in \Theta'(a)$ *and* $X \in \mathscr{U}(\mathfrak{g})$*, we get:*

$$\alpha_{\sigma(y)}(\sigma(X)) = \sigma(\alpha_y(X)),$$

where $\sigma : \mathrm{G} \to \mathrm{G}, y \to xyx^{-1}$ *that we extend to* $\mathscr{U}(\mathfrak{g})$*.*

Proof. According to Corollary 5.3.4, we get:

$$X = \alpha_y(X) + \Gamma_{ay}(\beta_y(X)). \tag{5.6}$$

Similarly, we get:

$$\sigma(X) = \alpha_{\sigma(y)}(X) + \Gamma_{a\sigma(y)}(\beta_{\sigma(y)}(X)). \tag{5.7}$$

By definition of $\mathrm{Z}(a)$, $a\sigma(y) = axya^{-1} = x(ay)x^{-1} = \sigma(ay)$. According to Lemma 5.4.4, we get:

$$\Gamma_{a\sigma(y)}(\beta_{\sigma(y)}(X)) = \Gamma_{\sigma(ay)}(\beta_{\sigma(y)}(X)) = \sigma(\Gamma_{ay}(\beta_y(X))). \tag{5.8}$$

In particular, using Equations (5.7) and (5.8), we get:

$$X = \sigma^{-1}(\sigma(X)) = \sigma^{-1}(\alpha_{\sigma(y)}) + \Gamma_{ay}(\beta_y(X)).$$

By Equation (5.6),

$$\sigma^{-1}(\alpha_{\sigma(y)}(X)) + \Gamma_{ay}(\beta_y(X)) = \alpha_y(X) + \Gamma_{ay}(\beta_y(X)).$$

In particular, $\sigma^{-1}(\alpha_{\sigma(y)}(X)) = \alpha_y(X)$ and we get:

$$\alpha_{\sigma(y)}(X) = \sigma(\alpha_y(X)).$$

□

5.4 The map δ

5.4.1 Local expression of a differential operator

Let G_1 be a nonempty open subset of G. The space $\mathscr{C}_c^\infty(\mathrm{G}_1)$ can be considered as a subset of $\mathscr{C}_c^\infty(\mathrm{G})$, i.e.

$$\mathscr{C}_c^\infty(\mathrm{G}_1) = \{\Psi \in \mathscr{C}_c^\infty(\mathrm{G}), \operatorname{supp}(\Psi) \subseteq \mathrm{G}_1\}.$$

Let $X \in \mathfrak{g}_0$. The element X defines a differential operator on G_1. Indedd, for $f \in \mathscr{C}_c^\infty(\mathrm{G}_1)$ and $x \in \mathrm{G}_1$, we get:

$$Xf(x) = \frac{d}{dt} f(x \exp(tX))_{|t=0}.$$

We know that $X, Y \in \mathfrak{g}_0, f \in \mathscr{C}_c^\infty(\mathrm{G}_1)$, we get:

$$[X, Y](f) = X(Yf) - Y(Xf).$$

We get a representation of $_0$ on the space $\mathscr{C}_c^\infty(\mathrm{G}_1)$ and then, a representation of $\mathscr{U}(\mathfrak{g})$ on $\mathscr{C}_c^\infty(\mathrm{G}_1)$. In particular, every element $b \in \mathscr{U}(\mathfrak{g})$ defines a differential operator on G_1.
We get the following lemma.

Lemma 5.4.1. *Let x_0 be a point of G_1 and b be an element of $\mathscr{U}(\mathfrak{g})$. If $bf(x_0) = 0$ for every $f \in \mathscr{C}_c^\infty(\mathrm{G}_1)$, then $b = 0$.*

Corollary 5.4.2. *Let D be a differential operator on G_1 and x_0 a point in G_1. Then, there exists a unique element $b \in \mathscr{U}(\mathfrak{g})$ such that*

$$D_{x_0} f := Df(x_0) = df(x_0)$$

for every $f \in \mathscr{C}_c^\infty(\mathrm{G}_1)$.

The proof of Lemma 5.4.1 and Corollary 5.4.2 can be found in [7, Section 4].

Definition 5.4.3. The element b (or D_{x_0}) is called the local expression of D at x_0 given in Corollary 5.4.2.

We get the following characterisation of invariants operators.

Lemma 5.4.4. *If D is a differential operator on G_1, the operator D is invariant if and only if for every $x \in \mathrm{G}$ and $y \in \mathrm{G}_1$, $D_{xyx^{-1}} = \mathrm{Ad}(x) D_y$.*

5.4.2 Transfer of differential operators

Let U_{G} be an open neighbourhood of $a \in \mathrm{G}$ and $\mathrm{U}_{\Theta(a)} = \Theta'(a) \cap a^{-1}\,\mathrm{U}_{\mathrm{G}}$. Because $1 \in \Theta'(a)$, we get that $\mathrm{U}_{\Theta(a)}$ is an open neighbourhood of 1 in $\Theta(a)$.

Definition 5.4.5. Let D be a differential operator on U_{G}.The operator $\Delta(\mathrm{G})$ defined locally by

$$\Delta(D)_y = \alpha_y(D_{ay}) \qquad (y \in \mathrm{U}_{\Theta(a)})$$

defined a differential operator on $\mathrm{U}_{\Theta(a)}$.

Lemma 5.4.6. *If both* U_{G} *and* D *are* G*-invariant, then* $\mathrm{U}_{\Theta(a)}$ *and* $\Delta(D)$ *are* $\mathrm{Z}(a)$*-invariant.*

Proof. Let $z \in \mathrm{Z}(a)$. Obviously, $z(a^{-1}\,\mathrm{U}_{\mathrm{G}})z^{-1} = a^{-1}(z\,\mathrm{U}_{\mathrm{G}}\,z^{-1}) = a^{-1}\,\mathrm{U}_{\mathrm{G}}$. Moreover, $z\Theta(a)z^{-1} = \Theta(a)$ and for every $y \in \Theta'(a)$, we get

$$\begin{aligned} \nu_a(zyz^{-1}) &= \det(\mathrm{Ad}(azyz^{-1})^{-1} - \mathrm{Id})_{\mathfrak{g}_0/\mathfrak{z}(a)} = \det(\mathrm{Ad}(z)^{-1}(\mathrm{Ad}(ay)^{-1} - \mathrm{Id})\,\mathrm{Ad}(z^{-1}))_{\mathfrak{g}_0/\mathfrak{z}(a)} \\ &= \det(\mathrm{Ad}(ay)^{-1} - \mathrm{Id})_{\mathfrak{g}_0/\mathfrak{z}(a)} = \nu_a(y) \neq 0. \end{aligned}$$

In particular, $z\Theta'(a)z^{-1} = \Theta'(a)$ and it follows that $\mathrm{U}_{\Theta(a)}$ is $\mathrm{Z}(a)$-invariant.
To prove that $\Delta(D)$ is $\mathrm{Z}(a)$-invariant, we are going to use Lemma 5.4.4, i.e. that $\Delta(D)_{xyx^{-1}} = \mathrm{Ad}(x)\Delta(D)_y$ for $y \in \mathrm{U}_{\Theta(a)}$ and $x \in \mathrm{Z}(a)$ (recall that $\mathrm{U}_{\Theta(a)} \subseteq \mathrm{Z}(a)$). We get:

$$\begin{aligned} \Delta(D)_{xyx^{-1}} &= \alpha_{xyx^{-1}}(D_{axyx^{-1}}) = \alpha_{xyx^{-1}}(D_{xayx^{-1}}) = \alpha_{xyx^{-1}}(\mathrm{Ad}(x)D_{ay}) \\ &= \alpha_{xyx^{-1}}(\mathrm{Ad}(x)D_{ay}) \qquad (\text{ D is G-invariant }) \\ &= \mathrm{Ad}(x)(\alpha_y(D_{ay})) \qquad (\text{Corollary 5.3.5}) \\ &= \mathrm{Ad}(x)\Delta(D)_y \end{aligned}$$

□

5.4.3 Definition and properties of δ

We denote by $\delta = \delta_{a,\mathrm{G}/\Theta(a)}$ the well-defined map:

$$\delta_{a,\mathrm{G}/\Theta(a)} : \mathrm{D}(\mathrm{U}_{\mathrm{G}}) \ni D \to \delta_{a,\mathrm{G}/\Theta(a)}(D) = \Delta(D) \in D(\mathrm{U}_{\Theta(a)}). \tag{5.9}$$

Let b be an element of $\mathrm{U}_{\Theta(a)}$ which is regular in $\Theta(a)$. We know that $\mathfrak{z}(a)$ is reductive and $\mathrm{rk}(\mathfrak{z}(a)) = \mathrm{rk}(\mathfrak{g}_0)$. Let $\mathfrak{z}(b)$ be the centraliser of b in $\mathfrak{z}(a)$. Then, $\mathfrak{z}(b)$ is a Cartan subalgebra of $\mathfrak{z}(a)$ (see Lemma 5.1.18) and because $\mathrm{rk}(\mathfrak{z}(a)) = \mathrm{rk}(\mathfrak{g}_0)$, $\mathfrak{z}(b)$ is a Cartan subalgebra of $\mathfrak{g}_0$.
Let $Z(b)$ be the centraliser of b in $\mathrm{Z}(a)$ and $\Theta(b) = \mathrm{Z}(b)_0$. We define $\mathrm{U}_{\Theta(b)}$ by:

$$\mathrm{U}_{\Theta(b)} = \left\{c \in \Theta(b), c \in b^{-q}\,\mathrm{U}_{\Theta(a)}, \det(\mathrm{Ad}(bc)^{-1} - \mathrm{Id})_{\mathfrak{z}(a)/\mathfrak{z}(b)} \neq 0\right\}.$$

Exercice 21. Prove that $c \in \mathrm{U}_{\Theta(b)}$ if and only if $c \in (ab)^{-1}\,\mathrm{U}_{\mathrm{G}}$ and $\det(\mathrm{Ad}(abc)^{-1} - \mathrm{Id})_{\mathfrak{g}_0/\mathfrak{z}(b)} \neq 0$. Moreover, $\mathrm{U}_{\Theta(b)}$ is $\Theta(b)$-invariant and $\mathfrak{z}(b)$ is the centraliser of ab in $\mathfrak{g}_0$.

By definition, $b \in \mathrm{U}_{\Theta(a)}$ and then $1 \in .\,\mathrm{U}_{\Theta(b)}$. It follows that $ab \in \mathrm{G}'$ (because $\det(\mathrm{Ad}(ab)^{-1} - \mathrm{Id})_{\mathfrak{g}_0/\mathfrak{z}(b)} \neq 0$).

Lemma 5.4.7. *Let $D \in \mathrm{D}(\mathrm{U}_{\mathrm{G}})$. Then,*

$$\delta_{b,\Theta(a)/\Theta(b)}\left(\delta_{a,\mathrm{G}/\Theta(a)}(\mathrm{D})\right) = \delta_{ab,\mathrm{G}/\Theta(b)}(D).$$

where the right-hand side is well defined because $\mathfrak{z}(b)$ is the centraliser of ab in $\mathfrak{g}_0$ (see Exercice 21).
Moreover, if $D \in \mathrm{D}_{\mathrm{G}}(\mathrm{U}_{\mathrm{G}})$, $\delta_{ab,\mathrm{G}/\Theta(b)} \in \mathrm{D}_{\mathrm{Z}(b)}(\mathrm{U}_{\Theta(b)})$.

5.5 A key lemma

5.5.1 A standard result of differential geometry

Let M and N be two oriented manifolds of dimension m and n respectively (we denote by ω_M and ω_N the corresponding m and n-forms on M and N). Let $\Phi : M \to N$ a surjective $\mathscr{C}^\infty$ mapping such that $\mathrm{rk}(d\Pi) = n$ everywhere.

Theorem 5.5.1. *Assume that ω_N is nowhere zero. Then, for every $\alpha \in \mathscr{C}_c^\infty(M)$, there exists a unique function $f_\alpha \in \mathscr{C}_c^\infty(M)$ such that:*

$$\int_M F \circ \Phi \alpha \omega_M = \int_N F f_\alpha \omega_N.$$

for all $F \in \mathscr{C}_c^\infty(N)$. Moreover, if ω_M is positive, then $\mathscr{C}_c^\infty(M) \ni \alpha \to f_\alpha \in \mathscr{C}_c^\infty(N)$ is surjective.

Remark 5.5.2. If $\omega_M > 0$, it defines a positive Borel measure μ_M on M such that for every $f \in \mathscr{C}_c^\infty(M)$,

$$\int_M f(m)d\mu_M(m) = \int_M f\omega_M.$$

In particular, if M and N are two Lie groups and μ_M and μ_N the corresponding Haar measures. Then, for every $\alpha \in \mathscr{C}_c^\infty(M)$, there exists $f_\alpha \in \mathscr{C}_c^\infty(N)$ such that

$$\int_M F \circ \Phi(m)\alpha(m)d\mu_M(m) = \int_N F(n)f_\alpha(n)d\mu_N(n). \tag{5.10}$$

We use the previous notations. If $f_1 \in \mathscr{C}_c^\infty(M)$ and $f_2 \in \mathscr{C}_c^\infty(N)$, we define a function $f_1 \otimes f_2$ on $M \times N$ by:

$$f_1 \otimes f_2(m,n) = f_1(m)f_2(n).$$

In particular, we get a linear mapping:

$$\mathscr{C}_c^\infty(M) \otimes \mathscr{C}_c^\infty(N) \to \mathscr{C}_c^\infty(M \times N).$$

Proposition 5.5.3. *The image of $\mathscr{C}_c^\infty(M) \otimes \mathscr{C}_c^\infty(N)$ is dense in $\mathscr{C}_c^\infty(M \times N)$.*

To finish, we recalled a last result about left-invariant distributions.

Lemma 5.5.4. *Let T be a distribution of an open connected subset U of G. Suppose $XT = 0$ for all $X \in \mathfrak{g}$. Then, T is constant.*

5.5.2 The distribution σ_T

Fix a semisimple element a of G.

Lemma 5.5.5. *Let $\Phi : \mathrm{G} \times \Theta(a) \to \mathrm{G}$ the map given by:*

$$\Phi(x, y) = x(ay)x^{-1}$$

The rank of the map Φ is of rank $\dim(\mathrm{G})$ *everywhere.*

Exercice 22. Prove Lemma 5.5.5.

Let U be an open neighbourhood of 1 in $\Theta'(a)$ which is $\Theta(a)$-invariant. We define by $\Omega = \Omega(U) = \Phi(\mathrm{G} \times \mathrm{U})$. In particular, because of the Lemma 5.5.5, $\Omega(\mathrm{U})$ is open in G.
According to Theorem 5.5.1, for every function $\alpha \in \mathscr{C}_c^\infty(\mathrm{G} \times \mathrm{U})$, there exists $f_\alpha \in \mathscr{C}_c^\infty(\Omega)$ such that for all $F \in \mathscr{C}_c^\infty(\Omega)$,

$$\int_{\mathrm{G} \times \mathrm{U}} F \circ \Phi(g, u)\alpha(g, u)dgdu = \int_\Omega f_\alpha(g)F(g)dg.$$

Lemma 5.5.6. *Let T be an invariant distribution on Ω. Then, there exists a unique distribution σ_T on U such that:*

$$T(f_\alpha) = \sigma_T(\beta_\alpha),$$

where $\alpha \in \mathscr{C}_c^\infty(\mathrm{G} \times \mathrm{U})$ and β_α is the function on U given by:

$$\beta_\alpha(y) = \int_\mathrm{G} \alpha(x, y)dx \qquad (y \in \mathrm{U}).$$

Moreover, σ_T is $\Theta(a)$-invariant and $\sigma_T = 0$ imply $T = 0$.

Proof. Because the map $\Phi : \mathrm{G} \times \mathrm{U} \to \Omega$ is surjective, we get using Theorem 5.5.1, the map:

$$\mathscr{C}_c^\infty(\mathrm{G} \times \mathrm{U}) \ni \alpha \to f_\alpha \in \mathscr{C}_c^\infty(\Omega), \tag{5.11}$$

is continuous and surjective and we define a linear map T' on $\mathscr{C}_c^\infty(\mathrm{G}\times\mathrm{U})$ by

$$T'(\alpha) = T(f_\alpha).$$

Because the map 5.11 is continuous, it follows easily that T' is a distribution on $\mathrm{G}\times\mathrm{U}$.
Let $x_0 \in \mathrm{G}$ and $\alpha \in \mathscr{C}_c^\infty(\mathrm{G}\times\mathrm{U})$. We denote by α_{x_0} the function of $\mathscr{C}_c^\infty(\mathrm{G}\times\mathrm{U})$ given by:

$$\alpha_{x_0}(x,y) = \alpha(x_0x,y).$$

Then, $T'(\alpha) = T'(\alpha_{x_0})$. Indeed, for every $F \in \mathscr{C}_c^\infty(\Omega)$, we get:

$$\begin{aligned}
&\int_\Omega f_{\alpha_{x_0}}(g)F(g)dg = \int_{\mathrm{G}\times\mathrm{U}} \alpha_{x_0}(x,y)F\circ\Phi(x,y)dxdy \\
= &\int_{\mathrm{G}\times\mathrm{U}} \alpha(x_0x,y)F(x(ay)x^{-1})dxdy = \int_{\mathrm{G}\times\mathrm{U}} \alpha(x,y)F(x_0^{-1}(x(ay)x^{-1})x_0^{-1})dxdy \\
= &\int_{\mathrm{G}\times\mathrm{U}} \alpha(x,y)F^{x_0}\circ\Phi(x,y)dxdy = \int_\Omega f_\alpha(g)F^{x_0}(g)dg \\
= &\int_\Omega f_\alpha(g)F(x_0gx_0^{-1})dg = \int_\Omega f_\alpha(x_0^{-1}gx_0)F(g)dg \\
= &\int_\Omega f_\alpha^{x_0^{-1}}(g)F(g)dg.
\end{aligned}$$

Then,

$$\int_\Omega \left(f_\alpha^{x_0^{-1}} - f_{\alpha_{x_0}}\right)(g)F(g)dg = 0 \qquad (F \in \mathscr{C}_c^\infty(\Omega)).$$

In particular, $f_\alpha^{x_0^{-1}} = f_{\alpha_{x_0}}$ and by the G-invariance of T, we get:

$$T'(\alpha_{x_0}) = T(f_{\alpha_{x_0}}) = T(f_\alpha^{x_0^{-1}}) = T(f_\alpha) = T'(\alpha).$$

For every $\beta \in \mathscr{C}_c^\infty(\mathrm{U})$, we define the distribution T'_β on G by:

$$T'_\beta(\gamma) = T'(\gamma\otimes\beta) \qquad (\gamma \in \mathscr{C}_c^\infty(\mathrm{G})).$$

This map is well-defined. According to Lemma 5.5.4, we get $T'_\beta = c(\beta) \in \mathbb{C}$, i.e. for every $\gamma \in \mathscr{C}_c^\infty(\mathrm{G})$,

$$T'_\beta(\gamma) = \int_\mathrm{G} c(\beta)\gamma(g)dg.$$

If γ is such that $\int_\mathrm{G} \gamma(g)dg = 1$, we get that:

$$c(\beta) = T\text{''}_\beta(\gamma) = T'(\gamma\otimes\beta).$$

In particular,

$$\tau_T : \mathscr{C}_c^\infty(\mathrm{U}) \ni \beta \to c(\beta) \in \mathbb{C},$$

is a distribution on U and then, for every $\alpha = \gamma \otimes \beta$, we get:

$$T'(\gamma \otimes \beta) = \sigma_T(\beta) \int_{\mathrm{G}} \gamma(g)dg = \sigma_T\left(\left(\int_{\mathrm{G}} \gamma(g)dg\right)\beta\right) = \sigma_T(\beta_\alpha).$$

Using Proposition 5.5.3, we get that:

$$T'(\alpha) = \sigma_T(\beta_\alpha) \qquad (\alpha \in \mathscr{C}_c^\infty(\mathrm{G} \times \mathrm{U}).$$

Let's now prove that the distribution σ_T is $\Theta(a)$-invariant, i.e. $\sigma_T(\beta) = \sigma_T(\beta^\xi), \beta \in \mathscr{C}_c^\infty(\mathrm{U}), \xi \in \Theta(a)$. Let $\alpha \in \mathscr{C}_c^\infty(\mathrm{G} \times \mathrm{U})$. We denote by $\alpha^{2,\xi}$ the function of $\mathscr{C}_c^\infty(\mathrm{G} \times \mathrm{U})$ given by:

$$\alpha^{2,\xi}(x, y) = \alpha(x, \xi^{-1} y \xi).$$

We get that $T'(\alpha) = T'(\alpha^{2,\xi})$. Indeed, as before, we prove easily that $f_{\alpha^{2,\xi}} = f_{\widetilde{\alpha}}$, where $\widetilde{\alpha}$ is given by $\widetilde{\alpha}(x, y) = \alpha(x\xi^{-1}, y)$. Then,

$$\begin{aligned}
T'(\alpha^{2,\xi}) &= T(f_{\alpha^{2,\xi}}) = T(f_{\widetilde{\alpha}}) = \sigma_T(\beta_{\widetilde{\alpha}}) \\
&= \sigma_T\left(y \to \int_{\mathrm{G}} \widetilde{\alpha}(x, y)dx\right) = \sigma_T\left(y \to \int_{\mathrm{G}} \alpha(x\xi^{-1}, y)dx\right) \\
&= \sigma_T\left(y \to \int_{\mathrm{G}} \alpha(x, y)dx\right) = \sigma_T(\beta_\alpha) = T(f_\alpha) = T'(f_\alpha).
\end{aligned}$$

Now, for every $\beta \in \mathscr{C}_c^\infty(\mathrm{U})$, we can choose a function $\alpha \in \mathscr{C}_c^\infty(\mathrm{G} \times \mathrm{U})$ such that $\beta = \beta_\alpha$. Then,

$$\begin{aligned}
\sigma_T(\beta^\xi) &= \sigma_T(\beta_\alpha^\xi) = \sigma_T\left(y \to \int_{\mathrm{G}} \alpha(x, \xi^{-1} y \xi)dx\right) \\
&= \sigma_T(\beta_{\alpha^{2,\xi}}) = T(f_{\alpha^{2,\xi}}) = T'(\alpha^{2,\xi}) \\
&= T'(\alpha) = \sigma_T(\beta_\alpha) = \sigma_T(\beta).
\end{aligned}$$

It implies that σ_T is $\Theta(a)$-invariant.

□

Exercice 23. With the notations of Lemma 5.5.6, prove that $\sigma_T = 0$ imply $T = 0$.

We get the following corollary.

Corollary 5.5.7. *Let D be a differential operator on Ω. Then, $\sigma_{DT} = \delta_a(D)\sigma_T$.*

5.6 Proof of the main theorem

5.6.1 Some isomorphisms

Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{m}$ be a reductive subalgebra of $\mathfrak{g}$ such that $\text{rk}(\mathfrak{m}) = \text{rk}(\mathfrak{g})$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{m}$. Because of our assumptions, $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathscr{W} = \mathscr{W}(\mathfrak{g}, \mathfrak{t})$ and $\mathscr{W}(\mathfrak{m}) = \mathscr{W}(\mathfrak{m}, \mathfrak{t})$ the corresponding Weyl. groups.
We denote by η^+ the subalgebra of $\mathfrak{g}$ given by $\eta^+ = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{t})} \mathbb{C}X_\alpha$, where $\mathfrak{g}_\alpha = \mathbb{C}X_\alpha$. Similarly, we denote by $\mathscr{N}$ and $\mathscr{P}$ the following subspaces of $\mathscr{U}(\mathfrak{g})$ given by:

$$\mathscr{N} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{t})} Y_\alpha \mathscr{U}(\mathfrak{g}) \qquad \mathscr{P} = \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{t})} X_\alpha \mathscr{U}(\mathfrak{g}).$$

where Y_α is a basis of $\mathfrak{g}_{-\alpha}$ (as in Proposition 2.2.5).

Lemma 5.6.1. *We get the following decomposition:*

$$\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{t}) \oplus (\mathscr{P} + \mathscr{N}). \tag{5.12}$$

We denote by $p_1 : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{t})$ the natural projection corresponding to Equation (5.12). We restrict this map to $\mathrm{Z}(\mathscr{U}(\mathfrak{g})$. We denote by ζ_1 the map:

$$\zeta_1 : \mathfrak{t} \ni h \to \zeta_1(h) = h - \rho(h).1 \in \mathrm{S}(\mathfrak{t}),$$

where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{t})} \alpha \in \mathfrak{t}^*$.
Using the universal property, we can extend the map ζ_1 to $\mathrm{S}(\mathfrak{t})$. We denote by γ the map:

$$\gamma = \zeta_1 \circ p_1 : \mathrm{Z}(\mathscr{U}(\mathfrak{g})) \to \mathrm{S}(\mathfrak{t}).$$

Theorem 5.6.2. *The map γ is an algebra homomorphism which is injective. Moreover,* $\text{Im}(\gamma) = \mathrm{S}(\mathfrak{t})^{\mathscr{W}}$ *and then:*

$$\gamma : \mathrm{Z}(\mathscr{U}(\mathfrak{g})) \to \mathrm{S}(\mathfrak{t})^{\mathscr{W}}. \tag{5.13}$$

is a bijection.

Similarly, we have a bijection $\gamma_{\mathfrak{m}} : \mathrm{Z}(\mathscr{U}(\mathfrak{m})) \to \mathrm{S}(\mathfrak{t})^{\mathscr{W}(\mathfrak{m})}$ and because $\mathscr{W}(\mathfrak{m}) \subseteq \mathscr{W}$, we get that $\mathrm{S}(\mathfrak{t})^{\mathscr{W}} \subseteq \mathrm{S}(\mathfrak{t})^{\mathscr{W}(\mathfrak{m})}$. In particular, we get a map $\mu_{\mathfrak{g}/\mathfrak{m}}$ given by:

$$\mu_{\mathfrak{g}/\mathfrak{m}} : \mathrm{Z}(\mathfrak{U}(\mathfrak{g})) \mapsto \mathrm{S}(\mathfrak{t})^{\mathscr{W}} \mapsto \mathrm{S}(\mathfrak{t})^{\mathscr{W}(\mathfrak{m})} \mapsto \mathrm{Z}(\mathscr{U}(\mathfrak{m})).$$

We get the following result.

Lemma 5.6.3. $\mathrm{Z}(\mathscr{U}(\mathfrak{m}))$ *is a free abelian module over* $\mu_{\mathfrak{g}/\mathfrak{m}}(\mathrm{Z}(\mathscr{U}(\mathfrak{g})))$ *of rank* $[\mathscr{W}, \mathscr{W}(\mathfrak{m})]$.

Remark 5.6.4. Obviously, $\mu_{\mathfrak{g}/\mathfrak{t}} = \gamma$ and in particular, $\mathrm{S}(\mathfrak{t})$ is a free $\mathrm{S}(\mathfrak{t})^{\mathscr{W}}$-abelian module of rank $|\mathscr{W}|$.

We have the following decomposition of the symmetric algebra $\mathrm{S}(\mathfrak{g})$:

$$\mathrm{S}(\mathfrak{g}) = \mathrm{S}(\mathfrak{t}) \oplus (\mathrm{S}(\mathfrak{g}\eta^{+} + \mathrm{S}(\mathfrak{g})\eta^{-}) . \tag{5.14}$$

We denote by $p_2 : \mathrm{S}(\mathfrak{g}) \to \mathrm{S}(\mathfrak{t})$ the projection corresponding to Equation (5.14).

Theorem 5.6.5 (Chevalley's isomorphism)**.** *The restriction of p_2 to $\mathrm{S}(\mathfrak{g})^{\mathrm{G}}$ is injective and we have* $\mathrm{Im}(p_2) = \mathrm{S}(\mathfrak{h})^{\mathscr{W}}$.

Using Theorems 5.6.2 and 5.6.5, we get an isomorphism:

$$p : \mathrm{Z}(\mathscr{U}(\mathfrak{g})z \to p(z) = p_2^{-1}(\gamma(z))\,\mathrm{S}(\mathfrak{g})^{\mathrm{G}}.$$

We denote by η the restriction of the map (5.2) to $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$:

$$\eta : \mathrm{Z}(\mathscr{U}(\mathfrak{g})) \to \mathrm{D}_{\mathrm{G}}^{\mathrm{G}}(\mathrm{G}). \tag{5.15}$$

Remark 5.6.6. Harish-Chandra's isomorphism classify all the possible infinitesimal character. Indeed, let $\lambda : \mathfrak{t} \to \mathbb{C}$ be a linear map. Using he universal property of the symmetric algebra [3, Appendix C], the linear form λ can be extended to a linear map $\lambda : \mathrm{S}(\mathfrak{t}) \to \mathbb{C}$ and by using the map λ, we get a map $\chi_\lambda : \mathrm{Z}(\mathscr{U}(\mathfrak{g})) \to \mathbb{C}$ given by:

$$\chi_\lambda(z) = \lambda(\gamma(z)) \qquad (z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))).$$

We recall the following theorem.

Theorem 5.6.7. *Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$. Then every homomorphism of $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ into $\mathbb{C}$ sending 1 to 1 is of the form χ_λ, $\lambda \in \mathfrak{t}^*$. If λ and $\lambda' \in \mathfrak{t}^*$, then $\chi_\lambda = \chi_{\lambda'}$ if and only if λ and λ' are in the same $\mathscr{W}$-orbit.*
In particular, $\mathrm{Spec}(\mathrm{Z}(\mathscr{U}(\mathfrak{g}))) \approx \mathfrak{t}^*/\mathscr{W}$.

The proof of this result can be found in [23].

5.6.2 A Theorem about some invariant distributions

For the moment, we admit the following results.

Lemma 5.6.8. *Let Ω be a completely invariant set of G and D be a G-invariant differential operator on Ω. Then, the following two conditions are equivalent:*

1. *$\delta_a(D) = 0$ for every regular element $a \in \Omega$,*

2. *For any open subset Ω_0 of Ω and. a locally invariant smooth function on Ω_0, $Df = 0$.*

Theorem 5.6.9. *Let Ω be a completely invariant open set in $\mathfrak{g}_0$ and T be an invariant distribution on Ω. Let D be an analytic and invariant differential operator on Ω such that $Df = 0$ for every invariant smooth function f on Ω. Then $DT = 0$.*

We now prove the following theorem.

Theorem 5.6.10. *Let Ω be a completely invariant open set in G and D be an analytic differential operator on Ω. Assume that:*

1. *D is invariant under G,*
2. *$\delta_{a,\mathrm{G}/\Theta(a)}(D) = 0$ for every regular element $a \in \Omega$.*

Then, $DT = 0$ for every invariant distribution T on Ω.

Proof. We prove that by induction on $\dim(\mathrm{G})$ We want to prove that $DT = 0$, i.e. $\mathrm{supp}(DT) = \emptyset$. Because Ω is completely invariant and $\mathrm{supp}(T)$ is G-invariant, we will prove that no semi-simple element of Ω lies in $\mathrm{supp}(T)$ (see Lemma 5.1.21).
Let a be a semisimple element in Ω. We distinguish two different cases.

1. $\underline{a \notin \mathrm{Z}(\mathrm{G})}$. Our goal here is to create a $\Theta(a)$-invariant distribution on a open neighbourhood of 1 in $\Theta(a)$ (completely invariant) satisfying the second condition of Theorem 5.6.10 for every regular b in $\Theta(a)$.

   Fix T an invariant distribution on Ω and consider the set $\mathrm{U}_{\Theta(a)} = \Theta'(a) \cap a^{-1}\Omega$. Obviously, this set is a completely invariant set of $\Theta(a)$. According to Lemma 5.5.6, there exists an invariant distribution σ_T on $\mathrm{U}_{\Theta(a)}$ (invariant under $\Theta(a)$ (or $\mathrm{Z}(a)$)). Moreover, $\sigma_{DT} = \delta_{a,\mathrm{G}/\Theta(a)}\sigma_T$.

   Fix b a regular element in $\Theta(a)$. According to Lemma 5.4.7, we get:

   $$\delta_{b,\Theta(a)/\Theta(b)}\left(\delta_{a,\mathrm{G}/\Theta(a)}(D)\right) = \delta_{ab,\mathrm{G}/\Theta(b)}(D) = 0$$

   according to our second assumption. In particular, $\delta_{a,\mathrm{G}/\Theta(a)}(D)$ is a $\mathrm{Z}(a)$-invariant differential operator on $\mathrm{U}_{\Theta(a)}$ (completely invariant space) such that $\delta_{b,\Theta(a)/\Theta(b)}\left(\delta_{a,\mathrm{G}/\Theta(a)}(D)\right) = 0$ for every b regular in $\Theta(a)$.

   Because $a \notin \mathrm{Z}(\mathrm{G})$, we have $\dim(\mathrm{Z}(a)) < \dim(\mathrm{G})$, we get by induction hypothesis that $\delta_{a,\mathrm{G}/\Theta(a)}(D)(H)$ for every $\Theta(a)$-invariant distribution H on $\mathrm{U}_{\Theta(a)}$. In particular, $\delta_{a,\mathrm{G}/\Theta(a)}(D)(\eta_T) = 0$. Using Corollary 5.5.7, we get that $0 = \delta_{a,\mathrm{G}/\Theta(a)}(D)(\eta_T) = \sigma_{DT} = 0$. It follows from Lemma 5.5.6 that $DT = 0$.

2. $\underline{a \in \mathrm{Z}(\mathrm{G})}$

Exercice 24. Prove that we can reduce the problem to the case $a = 1$, i.e. $1 \notin \text{supp}(DT)$.

The Lie algebra $\mathfrak{g}_0$ is reductive, in particular we get $\mathfrak{g}_0 = \mathrm{Z}(\mathfrak{g}_0) \oplus [\mathfrak{g}_0, \mathfrak{g}_0]$. Let $\mathfrak{c}_0$ be a relatively compact and open neighbourhood of 0 in $\mathrm{Z}(\mathfrak{g}_0)$ such that the restriction of exp to $\mathfrak{c}_0$ is injective.

According to Lemma 5.1.22, for every $0 < c \leq \pi$, we get that the restriction of exp to $[\mathfrak{g}_0, \mathfrak{g}_0](c)$ is injective. We denote by $\mathfrak{g}_1(c)$ the following subalgebra $\mathfrak{c}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_0](c)$.

Exercice 25. Prove that the restriction of the exponential map to. $\mathfrak{g}_1(c)$ is a diffeomorphism. Moreover, if we denote by Ω_0 the set $\Omega \cap \exp(\mathfrak{g}_1(c))$ and $\mathrm{U}_0 = log(\Omega_0)$, prove that U_0 is completely invariant.

For an invariant distribution T on Ω and let σ_T be the corresponding invariant distribution on U_0 (Lemma 5.5.6 and Section 5.1.3). Because D is an invariant operator on Ω_0, there exists a invariant differential operator $\xi(D)$ on U_0 (see Section 5.1.3).

Fix an invariant function ϕ on U_0. Because of our second assumption and using Lemma 5.6.8, we get $Df_\phi = 0$, where f_ϕ is defined in Equation (5.3). Using Equation 5.4, we get $Df_\phi = f_{\xi(D)}$.

In particular, $f_{\xi(D)} = 0$, i.e. $\xi(D) = 0$. According to Theorem 5.6.9, we get that $\xi(D)\eta_T = 0$ where η_T is defined in Equation 5.5. Using Lemma 5.1.24. we get that $\eta_{DT} = 0$, and we conclude by Lemma 5.5.6 that $DT = 0$. Which conclude the proof.

□

5.6.3 Proof

For every $X \in \mathfrak{g}$, we denote by $\partial(X)$ the differential operator on $\mathfrak{g}$ defined by:

$$\partial(X)\phi(Y) = \frac{d}{dt}_{|t=0} \phi(Y + tX),$$

where $\phi \in \mathscr{C}_c^\infty(\mathfrak{g})$, $Y \in \mathfrak{g}$, and then, we get a map:

$$\partial : \mathfrak{g} \to \mathrm{D}(\mathfrak{g})$$

which can be extended to $\mathrm{S}(\mathfrak{g})$. More generally, if U is an open subset of $\mathfrak{g}$, every element $X \in \mathfrak{g}$ define a differential operator on U.
We recall the bijective map $\eta : \mathrm{Z}(\mathscr{U}(\mathfrak{g})) \to \mathrm{D}_\mathrm{G}^\mathrm{G}(\mathrm{G})$. We get the following lemma.

Lemma 5.6.11. *For every locally invariant smooth function ϕ on an open subset V on U. Then,*

$$\eta(z)\phi = f_{\partial(p(z))\phi} \qquad (z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))). \tag{5.16}$$

Idea: According to Lemma 5.4, we already know that $\eta(z)\phi = f_{\xi(\eta(z))\phi}$. So we have to prove that $\eta(z)\phi = \delta(p(z))\phi$.

Corollary 5.6.12. *Assume that* U *is completely invariant. Then, if T is an invariant distribution on* $\mathrm{U_G}$,

$$\lambda_{\eta(z)T} = \partial(p(z))\lambda_T \qquad (z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))), \tag{5.17}$$

where λ_T is the corresponding distribution on U *defined by $\lambda_T(\phi) = T(f_\phi)$ (see Equation* (5.5)*)).*

Proof. Using Lemma 5.1.24 and Remark 5.1.25, we alreADY know that $\tau_{\eta(z)T} = \xi(\eta(z))\tau_T$. According to the "proof" of Lemma 5.6.11, we get:

$$(\xi(\eta(z)) - \partial(p(z)))\phi = 0, \tag{5.18}$$

for every invariant function ϕ. According to Theorem 5.6.9, we get that

$$(\xi(\eta(z)) - \partial(p(z)))\Lambda = 0, \tag{5.19}$$

for every invariant distribution Λ on U. Using Lemma 5.5.6), we get that τ_T is invariant and the result follows.

□

Theorem 5.6.13. *Let Ω be a completely invariant open set in* G *and T a distribution on Ω. We assume that*

1. *T is invariant,*
2. *There exists an ideal $\mathscr{U}$ in* $\mathrm{Z}(\mathscr{U}(\mathfrak{g}))$ *such that* $\dim_{\mathbb{C}} \mathrm{Z}(\mathscr{U}(\mathfrak{g}))/\mathscr{U} < \infty$ *and $\eta(u)T = 0$ for $u \in \mathscr{U}$.*

Then, T is given by a function $F_T \in \mathrm{L}^1_{loc}(\Omega)$ which is analytic on $\Omega' = \Omega \cap \mathrm{G}'$.

Let Ω_0 be the set of $a \in \Omega$ such that there exists an open neighbourhood U_a of a in Ω and a locally integrable function F_a on U_a such that $F_{a|_{\mathrm{U}_a \cap \mathrm{G}'}}$ is analytic and $T = F_a$ on U_a.

Lemma 5.6.14. *If $\Omega = \Omega_0$, the Theorem 5.6.13 holds.*

Proof. Let $a, b \in \Omega_0$ and (U_a, F_a) and (U_b, F_b) the corresponding data such that $\mathrm{U}_a \cap \mathrm{U}_b \neq$. Because F_a and F_b are analytic , then $F_a - F_b$ is analytic on $\mathrm{U}'_a \cap \mathrm{U}_b$ and

$$\int_{\mathrm{G}} (F_a - F_b)(g) f(g) dg = 0$$

for every $f \in \mathscr{C}_c^\infty(\mathrm{G})$ such that $\mathrm{supp}(f) \subseteq \mathrm{U}'_a \cap \mathrm{U}_b$. It implies that $F_a = F_b$ on $\mathrm{U}'_a \cap \mathrm{U}_b$, and then, we get a function F locally integrable on Ω analytic on Ω'.

□

Lemma 5.6.15. *The space Ω_0 is* G*-invariant.*

Proof. Let $a \in \Omega_0$ and (U_a, F_a) the corresponding data. Let $g \in \mathrm{G}$ and $U_{gag^{-1}} = L_g \circ R_{g^{-1}}(\mathrm{U}_a)$. We consider $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$ such that $\mathrm{supp}(\Psi) \subseteq \mathrm{U}_{gag^{-1}}$. Then, $\mathrm{supp}(\Psi \circ L_{g^{-1}} \circ R_g) \subseteq \mathrm{U}_a$ and because T is G-invariant, we get:

$$\begin{aligned} T(\Psi) &= T(\Psi \circ L_{g^{-1}} \circ R_g) = \int_{\mathrm{G}} \Psi \circ L_{g^{-1}} \circ R_g)(x) F_a(x) dx = \int_{\mathrm{G}} \Psi(g^{-1}xg) F_a(x) dx \\ &= \int_{\mathrm{G}} \Psi(x) F_a(gxg^{-1}) dx = T_{F_a \circ L_g \circ R_{g^{-1}}}(\Psi). \end{aligned}$$

□

Corollary 5.6.16. *The space $\Omega \setminus \Omega_0$ is* G*-invariant.*

Proof. Let $u \in \Omega \setminus \Omega_0$ and $g \in \mathrm{G}$. If $gug^{-1} \in \Omega_0$, then $u \in g^{-1}\Omega g = \Omega_0$, which is impossible. In particular, $g(\Omega \setminus \Omega_0)g^{-1} = \Omega \setminus \Omega_0$.

□

Remark 5.6.17. The space Ω_0 is open in Ω, so $\Omega \setminus \Omega_0$ is closed and G-invariant. According to Lemma 5.1.21, to prove that $\Omega = \Omega_0$, it is enough to prove that all semisimple elements of Ω lies in Ω_0.

We first prove the following lemma.

Lemma 5.6.18. $\Omega' \subseteq \Omega_0$.

In order to prove the previous lemma, we will admit some results.

Theorem 5.6.19. *Let Ω be a completely invariant open set in $\mathfrak{g}_0$ and T be an invariant distribution on Ω. Let D be an analytic and invariant differential operator on Ω such that $Df = 0$ for every invariant smooth function f on Ω. Then, $DT = 0$.*

Remark 5.6.20 (Remark before proof).

1. Let a be a semisimple element of $\Omega' = \Omega \cap \mathrm{G}'$ and $\mathfrak{z}(a)$ be the centraliser of a in $\mathfrak{g}_0$. We denote by $\Theta(a)$ the analytic subgroup corresponding to $\mathfrak{z}(a)$ and $\Theta'(a) = \Theta(a) \cap a^{-1}\mathrm{G}$ the set of $h \in \Theta(a)$ such that $\nu_a(h) \neq 0$, where $\nu_a(h) = \det(\mathrm{Ad}(ah)^{-1} - \mathrm{Id})_{\mathfrak{g}_0/\mathfrak{z}(a)}$. We get the following equality:

$$\delta_{a,\mathrm{G}/\Theta(a)}(\eta(z)) = |\nu_a|^{-\frac{1}{2}} \partial(\gamma(z)) \circ |\nu_a|^{\frac{1}{2}} \qquad (z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))). \tag{5.20}$$

   The proof of this reuslt can be found in [12].

2. Let $a \in \Omega'$ and $\mathrm{U}_{\Theta(a)} = \Theta'(a) \cap a^{-1}\Omega$. We consider the map ϕ:

$$\phi : \mathrm{G} \times \mathrm{U}_{\Theta(a)} \ni (g, y) \to g(ay)g^{-1} \in \mathrm{G}\,.$$

The map ϕ is everywhere regular, and we get a natural map:

$$\widetilde{\phi} : \mathrm{G}/\Theta(a) \times \mathrm{U}_{\Theta(a)} \to \mathrm{G},$$

which is regular everywhere. We fix $h_0 \in \mathrm{U}_{\Theta(a)}$. Using the constant rank theorem, one can prove that we can find an open neighbourhood G_0^* of 1 in $\mathrm{G}/\Theta(a)$ and U_a of h_0 in $\mathrm{U}_{\Theta(a)}$ such that

$$\widetilde{\phi}_{|_{\mathrm{G}_0^* \times \mathrm{U}_a}} : \mathrm{G}_0^* \times \mathrm{U}_a \to \phi(\widetilde{G_0^* \times \mathrm{U}_a})$$

is a diffeomorphism.

Proof of Lemma 5.6.18. Let a be an element of Ω' and $\mathfrak{z})a)$ the centraliser of $a \in \mathfrak{g}_0$ (which is a Cartan subalgebra, see Lemma 5.1.18). We denote by $\Omega_{\Theta(a)}$ the subset of $\Theta(a)$ given by $a^{-1}\Omega \cap \Theta'(a)$. According to Lemma 5.5.6, there exists a distribution σ_T on $\Omega_{\Theta(a)}$ satisfying $T(f_\alpha) = \sigma_T(\beta_\alpha)$ with $\mathscr{C}_c^\infty(\mathrm{G} \times \Omega_{\Theta(a)}) \ni \alpha \to f_\alpha \in \mathscr{C}_c^\infty(\phi(\mathrm{G} \times \Omega_{\Theta(a)}))$.
We denote by $\sigma = |\nu_a|^{\frac{1}{2}}\sigma_T$. According to Equation (5.20), we have:

$$\delta_{a,\mathrm{G}/\Theta(a)}(\eta(z))\sigma_T = |\nu_a|^{-\frac{1}{2}}\partial(\gamma(z))\sigma \qquad (z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g})))$$

and using Corollary 5.5.7, we get

$$\delta_{a,\mathrm{G}/\Theta(a)}(\eta(z))\sigma_T = \sigma_{\eta(z)T} \qquad (z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))).$$

In particular,

$$\sigma_{\eta(z)T} = |\nu_a|^{-\frac{1}{2}}\partial(\gamma(z))\sigma \qquad (z \in \mathrm{Z}(\mathscr{U}(\mathfrak{g}))).$$

By assumption, $\eta(z)T = 0$ for every $z \in \mathscr{U}$. In particular, $\delta(\gamma(z)) = 0$ for all $z \in \mathscr{U}$. Using Remark 5.6.4, $\mathscr{B} = \mathrm{S}(\mathfrak{t})\gamma(\mathscr{U})$ has finite co-dimension in $\mathrm{S}(\mathfrak{t})$.
Fix a basis $H_1, \ldots, H_r$ be a basis of $\mathfrak{t}$ over $\mathbb{R}$ and fix Δ the element of $\mathscr{U}(\mathfrak{t}) = \mathrm{S}(\mathfrak{t})$. We denote by $N = \dim_{\mathbb{C}} \mathrm{S}(\mathfrak{t})/\mathscr{B}$. Obviously, we can choose elements $c_i \in \mathbb{C}$, $1 \leq i \leq N$ such that

$$\widetilde{\Delta} = \Delta^N + \sum_{i=1}^{N} c_i \Delta^{N-i} \in \mathscr{B}.$$

Exercice 26. Prove that the differential operator $\partial(\widetilde{\Delta})$ is an analytic differential operator on $\Theta(a)$ such that $\partial(\widetilde{\Delta})\sigma = 0$. In particular, prove that σ coincides with an analytic function g on $\Omega_{\Theta(a)}$.

Let G_0^* the open neighbourhood of $1*$ in $\mathrm{G}/\Theta(a)$ and $\mathrm{U}_{\Theta(a)}$ the open neighbourhood of 1 in $\Omega_{\Theta(a)}$ such that $\widetilde{\phi} : \mathrm{G}_0^* \times \mathrm{U}_{\Theta(a)} \to \widetilde{\phi}(\mathrm{G}_0^* \times \mathrm{U}_{\Theta(a)}) := \widetilde{\mathrm{U}_{\Theta(a)}}$ is a diffeomorphism (Remark 5.6.20).
In particular, the map:

$$\mathscr{C}_c^\infty(\mathrm{G}_0^* \times \mathrm{U}_{\Theta(a)}) \ni \alpha \to f_\alpha \to \mathscr{C}_c^\infty(\widetilde{\mathrm{U}_{\Theta(a)}})$$

is surjective (see Theorem 5.5.1).
Using Lemma 5.5.6, $T(f_\alpha) = \sigma_T(\beta_\alpha)$ and according to Exercice 26, σ_T is given by the function $|\nu_a|^{-\frac{1}{2}}|g$ on $\widetilde{\mathrm{U}_{\Theta(a)}}$ (open neighbourhood of a in Ω). Then,

$$\begin{aligned} T(f_\alpha) &= \sigma_T(\beta_\alpha) = \int_{\mathrm{U}_{\Theta(a)}} \beta_\alpha(v)|\nu_a(v)|^{-\frac{1}{2}} g(v)dv \\ &= \int_{\mathrm{U}_{\Theta(a)}} \left(\int_{\mathrm{G}_0^*} \alpha(x^*, v)dx^* \right) |\nu_a(v)|^{-\frac{1}{2}} g(v)dv \\ &= \int_{\mathrm{G}_0^*} \int_{\mathrm{U}_{\Theta(a)}} \alpha(x^*, v) F_a \circ \widetilde{\phi}(x^*, v)dvdx^* \end{aligned}$$

where $F_a \circ \widetilde{\phi}(x^*, v) = |\nu_a(v)|^{-\frac{1}{2}} g(v)$. So it defines a function F_a on $\widetilde{\mathrm{U}_{\Theta(a)}}$ which is analytic on $\widetilde{\mathrm{U}_{\Theta(a)}}$. Then,

$$T(f_\alpha) = \int_{\mathrm{G}_0^*} \int_{\mathrm{U}_{\Theta(a)}} \alpha(x^*, v) F_a \circ \widetilde{\phi}(x^*, v)dvdx^* = \int_{\widetilde{\mathrm{U}_{\Theta(a)}}} f_\alpha(x) F_a(x)dx = T_{F_a}(f_\alpha).$$

In particular, $T = F_a$ on $\widetilde{\mathrm{U}_{\Theta(a)}}$ and $\widetilde{\mathrm{U}_{\Theta(a)}}$ is an open neighbourhood of a in Θ. In particular, $a \in \Omega_0$.

□

We get directly the following Corollary.

Corollary 5.6.21. *There exists an analytic function F on Ω'.*

We are now able to prove Theorem 5.6.13.

Proof of Theorem 5.6.13. We will give the main steps of the proof. Let a be a semisimple element of Ω. As in proof of Theorem 5.6.10, we will distinguish two cases:

1. $a \notin \mathrm{Z}(\mathrm{G})$. We will prove that by induction on $\dim(\mathrm{G})$ as in the proof of Theorem 5.6.10. In particular, $\dim(\mathfrak{z}(a)) < \dim(\mathfrak{g}_0)$. We denote by $\Omega_{\Theta(a)}$ the space $\Omega_{\Theta(a)} = a^{-1}\Omega \cap \Theta'(a)$: it is an open and completely invariant neighbourhood of 1 in $\Theta(a)$.

   (a) According to Lemma 5.5.6, there exists a distribution σ_T on $\Omega_{\Theta(a)}$ invariant under $\Theta(a)$.

   (b) We prove that $\mu_{\mathfrak{g}_0/\mathfrak{z}(a)}(u)\sigma = 0$ for every $u \in \mathscr{U}$.

   (c) According to Lemma 5.6.3, $\mathrm{Z}(\mathscr{U}(\mathfrak{z}(a)))$ is a finite module over $\mu_{\mathfrak{g}/\mathfrak{z}(a)}(\mathrm{Z}(\mathscr{U}(\mathfrak{g})))$, we get that $\mathscr{B} = \mathrm{Z}(\mathscr{U}(\mathfrak{g}))\mu_{\mathfrak{g}/\mathfrak{z}(a)}(\mathrm{Z}(\mathscr{U}(\mathfrak{g})))$ has finite codimension in $\mathrm{Z}(\mathscr{U}(\mathfrak{z}(a)))$.

   (d) As in the proof of Theorem 5.6.10, the distribution $\sigma = |\nu_a|^{\frac{1}{2}}\sigma_T$ is given by a locally integrable function on $\Omega_{\Theta(a)}$ which is analytic on $\Omega'_{\Theta(a)}$ (induction hypothesis).

   (e) We denote by $\phi : \mathrm{G} \times \Omega_{\Theta(a)} \ni (x, y) \to x(ay)x^{-1} \in \Omega$, $\mathrm{U} = \phi(\mathrm{G} \times \Omega_{\Theta(a)})$, $\mathrm{U}' = \mathrm{U} \cap \mathrm{G}' = \phi(\mathrm{G} \times \Omega'_{\Theta(a)}$.

(f) $T = F$ on Ω'. It implies that

$$T(f_\alpha) = \int_{G\times\Omega'_{\Theta(a)}} \alpha(x,y)F\circ\phi(x,y)dxdy$$

for every $\alpha \in \mathscr{C}_c^\infty(G\times\Omega'_{\Theta(a)}$. According to Lemma 5.5.6, $T(f_\alpha) = \sigma_T(\beta_\alpha)$ and then,

$$T(f_\alpha) = \int_{G\times\Omega'_{\Theta(a)}} \alpha(x,y)|\nu_a(y)|^{-\frac{1}{2}}g(y)dxdy.$$

It implies that $(x,y) \to F\circ\phi(x,y) - |\nu_a(y)|^{-\frac{1}{2}}g(y)$ is zero on $G\times\Omega'_{\Theta(a)}$ and then, we prove that $F\circ\phi$ is locally integrable on $G\times\Omega_{\Theta(a)}$. In particular, F is integrable on U.

(g) We get:

$$\begin{aligned}\int_U f_\alpha(x)F(x)dx &= \int G\times\Omega_{\Theta(a)}\alpha(x,y)F\circ\phi(x,y)dxdy \\ &= \int G\times\Omega_{\Theta(a)}\alpha(x,y)|\nu_a(y)|^{-\frac{1}{2}}g(y)dxdy \\ &= \sigma_T(\beta_\alpha) = T(f_\alpha)\end{aligned}$$

In particular, $T = F$ on $U = \phi(G\times\Omega_{\Theta(a)})$ and then, $a \in \Omega_0$.

2. $\underline{a \in Z(G)}$ Proof is similar to the one of Theorem 5.6.10.

□

Corollary 5.6.22. *The global character Θ_Π of an irreducible quasi-simple representation $(\Pi, \mathscr{H})$ is given by a locally integrable function F_Π on G which is analytic on G′.*

Proof. Take $T = \Theta_\Pi$, $\Omega = G$ and $\mathscr{U} = \mathrm{Ker}(\chi_\Pi)$ in Theorem 5.6.13.

□

We finish this section by an easy consequence of the previous lemma.

Lemma 5.6.23. *Let $(\Pi_1, \mathscr{H}_1)$ and $(\Pi_1, \mathscr{H}_1)$ two irreducible unitary representations of G. Then, Π_1 and Π_2 are equivalent if and only if $F_{\Pi_1|_{G'}} = F_{\Pi_2|_{G'}}$.*

Proof. Using Corollary 4.5.8, we already know Π_1 and Π_2 are equivalent if and only if $\Theta_{\Pi_1} = \Theta_{\Pi_2}$. What we are going to prove is that $\Theta_{\Pi_1} = \Theta_{\Pi_2}$ if and only if $F_{\Pi_1|_{G'}} = F_{\Pi_2|_{G'}}$.
Assume that there exists a locally integrable function $\tau_{\Pi_1} : G \to \mathbb{C}$ analytic on G′ such that

$$\Theta_\Pi(\Psi) = \int_G \Psi(g)\tau_{\Pi_1}(g)dg.$$

We get

$$\int_{\mathrm{G}} (F_{\Pi_1} - \tau_{\Pi_1})(g)\Psi(g)dg = 0 \qquad (\Psi \in \mathscr{C}_c^\infty(\mathrm{G})),$$

and in particular,

$$\int_{\mathrm{G}} (F_{\Pi_1} - \tau_{\Pi_1})(g)\Psi(g)dg = 0 \qquad (\Psi \in \mathscr{C}_c^\infty(\mathrm{G}')). \tag{5.21}$$

Because $F_{\Pi_1} - \tau_{\Pi_1} : \mathrm{G}' \to \mathbb{C}$ is analytic, for every $g' \in \mathrm{G}'$, we can find a Dirac sequence $\{f_n\}_n$ such that:

$$(F_{\Pi_1} - \tau_{\Pi_1})(g') = \lim_{n \to +\infty} \int_{\mathrm{G}'} (F_{\Pi_1} - \tau_{\Pi_1})(g) f_n(g) dg.$$

Using Equation (5.21), we get that $F_{\Pi_1} = \tau_{\Pi_1}$ on G'.
Conversely, if $F_{\Pi_1} - \tau_{\Pi_1} = 0$ on G'. In particular, for every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$,

$$\int_{\mathrm{G}} (F_{\Pi_1} - \tau_{\Pi_1})(g)\Psi(g)dg = 0,$$

i.e. $\Theta_\Pi = T_{F_{\Pi_1}} = T_{\tau_{\Pi_1}}$. It means that Θ_{Π_1} is parametrised by the restriction of F_{Π_1} to G'. Which prove the result. □

Chapter 6

Well-known facts about characters

6.1 A paper of Rossmann

Through this section, G will be a semisimple connected Lie group with a maximal compact subgroup K such that $\mathrm{rk}(\mathrm{K}) = \mathrm{rk}(\mathrm{G})$. We fix a compact Cartan subgroup of K (which is by assumption on the ranks a Cartan subgroup of G) and let $\mathfrak{t}_0, \mathfrak{k}_0, \mathfrak{g}_0$ the Lie algebra of $\mathrm{T}, \mathrm{K}, \mathrm{G}$ and let's denote by $\mathfrak{t}, \mathfrak{k}, \mathfrak{g}$ the complexifications. We denote by Π the map on $\mathfrak{t}_0$ given by:

$$\Pi : \mathfrak{t}_0 \ni t \to \prod_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} \alpha(t) \in \mathbb{C}.$$

and denote by $\mathfrak{t}_0^{\mathrm{reg}} = \{t \in \mathfrak{t}_0, \Pi(t) \neq 0\}$.
As before, we denote by $\mathscr{W} = \mathscr{W}(\mathfrak{g})$ the Weyl group corresponding to $(\mathfrak{g}, \mathfrak{t})$ and we denote by $\mathscr{W}(\mathrm{T})$ the following quotient $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$.

Remark 6.1.1. The group $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$ acts naturally on T by:

$$N_{\mathrm{G}}(\mathrm{T})/\mathrm{T} \times \mathrm{T} \ni (\bar{g}, t) \to gtg^{-1} \in \mathrm{T}.$$

This action is well-defined. Indeed, let $g, g' \in N_{\mathrm{G}}(\mathrm{T})$ such that $\bar{g} = \bar{g'}$. There exists $t' \in \mathrm{T}$ such that $g = g't'$ and then for $t \in \mathrm{T}$, we get:

$$gtg^{-1} = g't'tt'^{-1}g'^{-1} = g'tg'^{-1}.$$

In particular, we get a natural action of $\mathscr{W}(\mathrm{T})$ on $\mathfrak{t}_0$ and $\mathfrak{t}$, and then every element of $\mathscr{W}(\mathrm{T})$ defines an element of $\mathrm{GL}(\mathfrak{t})$.

Lemma 6.1.2. *The group $\mathscr{W}(\mathrm{T})$ is a subgroup of $\mathscr{W}(\mathfrak{g})$. Moreover, if G is compact and connected, $\mathscr{W}(\mathrm{T}) = \mathscr{W}(\mathfrak{g})$.*

Notation 6.1.3. We denote by dX and dt the Lebesgue measure on $\mathfrak{g}_0$ and $\mathfrak{t}_0$ respectively.

We now recall two basic Lemmas.

Lemma 6.1.4. *The map* $\phi : \mathrm{G}/\mathrm{T}\times\mathfrak{t}_0^{\mathrm{reg}} \to \mathrm{G}.\mathfrak{t}_0^{\mathrm{reg}} = \mathrm{Ad}(\mathrm{G})(\mathfrak{t}_0^{\mathrm{reg}})$ *is proper and has bijective differential everywhere. Moreover, the fibers of* ϕ *at* $g.t$ *is:*

$$\left\{(g\omega\,\mathrm{T}, \omega^{-1}t), \omega \in \mathscr{W}(\mathrm{T})\right\}.$$

Let $\overline{dg}$ be an Ad-invariant measure on G/T.

Lemma 6.1.5. *Let* f *be a measurable function on* $\mathrm{G}.\mathfrak{t}_0^{\mathrm{reg}}$. *Then,* $|f|$ *in integrable on* $\mathrm{G}.\mathfrak{t}_0^{\mathrm{reg}}$ *with respect to* dX *if and only if* $(\bar{g}, t) \to |\Pi(t)|^2|f(\bar{g}.t)|$ *is integrable on* $\mathrm{G}/\mathrm{T}\times\mathfrak{t}_0^{\mathrm{reg}}$ *with respect to* $\overline{dg}dt$. *Moreover, for every* $f \in \mathscr{C}_c^\infty(\mathfrak{g}_0)$ *such that* $\mathrm{supp}(f) \subseteq \mathrm{G}.\mathfrak{t}_0^{\mathrm{reg}}$,

$$\int_{\mathrm{G}.\mathfrak{t}_0^{\mathrm{reg}}} f(X)dX = \int \mathfrak{t}_0^{\mathrm{reg}}|\Pi(t)|^2 \int_{\mathrm{G}/\mathrm{T}} f(\mathrm{Ad}(g)t)\overline{dg}dt. \tag{6.1}$$

Proof. The proof of Lemma 6.1.8 can be found in [26, Lemma 1, Page 34] and for Lemma 6.1.5, one can check [26, Lemma 2, Page 35].

□

Remark 6.1.6. Let dg be a Haar neasure on G. Then, for $H \in \mathfrak{t}_0$,

$$\begin{aligned}\int_{\mathrm{G}} f(\mathrm{Ad}(g)H)dg &= \int_{\mathrm{G}/\mathrm{T}}\int_{\mathrm{T}} f(\mathrm{Ad}(gt)H)dt\overline{dg} = \int_{\mathrm{G}/\mathrm{T}}\int_{\mathrm{T}} f(\mathrm{Ad}(g)\,\mathrm{Ad}(t)H)dt\overline{dg}\\ &= \int_{\mathrm{G}/\mathrm{T}}\int_{\mathrm{T}} f(\mathrm{Ad}(g)H)dt\overline{dg} = \int_{\mathrm{G}/\mathrm{T}} f(\mathrm{Ad}(g)H)\overline{dg}\end{aligned}$$

In particular, Equation (6.1) can be written as:

$$\int_{\mathrm{G}.\mathfrak{t}_0^{\mathrm{reg}}} f(X)dX = \int \mathfrak{t}_0^{\mathrm{reg}}|\Pi(t)|^2 \int_{\mathrm{G}} f(\mathrm{Ad}(g)t)dgdt. \tag{6.2}$$

We use Rossmann's notations (see [25]). For every function ϕ on $\mathfrak{g}_0$, we denote by $M_{\mathfrak{t}_0}\phi$ the (partially defined) function on $\mathfrak{t}_0$ given by;

$$M_{\mathfrak{t}_0}\phi(t) = \Pi(t)\int_{\mathrm{G}} \phi(\mathrm{Ad}(g)t)dg \qquad (t \in \mathfrak{t}_0^{\mathrm{reg}}).$$

Exercice 27. Prove that if $\phi \in \mathscr{C}_c^\infty(\mathrm{G}.\mathfrak{t}_0^{\mathrm{reg}})$, then $M_{\mathfrak{t}_0}\phi \in \mathscr{C}_c^\infty(\mathfrak{t}_0^{\mathrm{reg}})$. Moreover, for $w \in \mathscr{W}(\mathrm{T}), t \in \mathfrak{t}_0^{\mathrm{reg}}$, $M_{\mathfrak{t}_0}\phi(w^{-1}t) = \varepsilon(w)M_{\mathfrak{t}_0}\phi(t)$.

We denote by $F_{\mathfrak{g}_0}$ and $F_{\mathfrak{t}_0}$ the Fourier transform on $\mathfrak{g}_0$ and $\mathfrak{t}_0$ respectively, i.e.

$$F_{\mathfrak{g}_0}(X) = \int_{\mathfrak{g}_0} e^{i\mathscr{K}(X,Y)}\phi(Y)dY \qquad (\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0), X \in \mathfrak{g}_0),$$

$$F_{\mathfrak{t}_0}(X) = \int_{\mathfrak{t}_0} e^{i\mathscr{K}(X,Y)}\phi(Y)dY \qquad (\phi \in \mathscr{C}_c^\infty(\mathfrak{t}_0), X \in \mathfrak{t}_0).$$

We fix $t \in \mathfrak{t}_0^{\text{reg}}$. We define the distribution μ_t on $\mathfrak{g}_0$ by:

$$\mu_t(\phi) = \Pi(t)\int_{\mathrm{G}} \phi(\mathrm{Ad}(g)t)dg \qquad (\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0)).$$

We denote by $\hat{\mu}_t = F_{\mathfrak{g}_0}\mu_t$ the Fourier transform of μ_t, i.e. for every $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0)$,

$$\hat{\mu}_t(\phi) = \mu_t(F_{\mathfrak{g}_0}\phi) = \Pi(t)\int_{\mathrm{G}} F_{\mathfrak{g}_0}\phi(\mathrm{Ad}(g)t)dg = \Pi(t)\int_{\mathrm{G}}\int_{\mathfrak{g}_0} e^{i\mathscr{K}(\mathrm{Ad}(g)t,X)}\phi(X)dXdg.$$

We admit for the moment the following result of Harish-Chandra.

Theorem 6.1.7. *The distribution $\hat{\mu}_t$ is an invariant distribution given by a locally integrable function $\hat{\mu}_t$ on $\mathfrak{g}_0$ whose restriction to $\mathfrak{t}_0^{\text{reg}}$ is given by:*

$$\hat{\mu}_t(s) = \Pi(s)^{-1}\sum_{w\in\mathscr{W}(\mathfrak{g})} c(w,t)e^{i\mathscr{K}(wt,s)}.$$

Moreover, for a fixed $w \in \mathscr{W}(\mathfrak{g})$, $t \to c(w,t)$ is a locally constant function on $\mathfrak{t}_0^{\text{reg}}$.

We now prove the following lemma.

Lemma 6.1.8. *1. The constants $c(w,t)$ are independant of t.*

2. The constants c satisfy:

(a) $c(w) = c(w^{-1})$ for every $w \in \mathscr{W}(\mathfrak{g})$,

(b) $c(vw) = \varepsilon(v)c(w)$ for $v \in \mathscr{W}(\mathrm{T})$ and $w \in \mathscr{W}(\mathfrak{g})$.

Remark 6.1.9. For $t \in \mathfrak{t}_0^{\text{reg}}$. Then,

$$|\Pi(t)|^2 = \Pi(t)\overline{P(t)} = \prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{t})}\alpha(t)\overline{\prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{t})}\alpha(t)} = \prod_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{t})}\alpha(t)\overline{\alpha(t)}.$$

Because $t \in \mathfrak{t}_0$, $\alpha(t) \in i\mathbb{R}$ and then $\overline{\alpha(t)} = -\alpha(t)$. In particular,

$$|\Pi(t)|^2 = (-1)^{|\Phi^+(\mathfrak{g},\mathfrak{t})|}\Pi(t).$$

Because of Equation (2.5), we get: $\dim_{\mathbb{C}}(\mathfrak{g}) = \dim_{\mathbb{C}}(\mathfrak{t}) + |\Phi^+(\mathfrak{g},\mathfrak{t})|$, and then, $|\Phi^+(\mathfrak{g},\mathfrak{t})| = \frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}$.

Proof of Lemma 6.1.8. To prove the first part of Lemma 6.1.8, we are going to prove that:

$$\Pi(s)\hat{\mu}_t(s) = \Pi(t)\hat{\mu}_s(t) \qquad (s,t \in \mathfrak{t}_0^{\text{reg}}). \tag{6.3}$$

Indeed, if (6.3) holds, then

$$\begin{aligned}\sum_{w\in\mathscr{W}(\mathfrak{g})} c(w,s)e^{i\mathscr{K}(ws,t)} &= \sum_{w\in\mathscr{W}(\mathfrak{g})} c(w,t)e^{i\mathscr{K}(wt,s)} = \sum_{w\in\mathscr{W}(\mathfrak{g})} c(w,t)e^{i\mathscr{K}(t,w^{-1}s)}\\ &= \sum_{w\in\mathscr{W}(\mathfrak{g})} c(w^{-1},s)e^{i\mathscr{K}(ws,t)}\end{aligned}$$

Fix $t\in\mathfrak{t}_0^{\mathrm{reg}}$. The functions $s\to e^{i\mathscr{K}(ws,t)}, w\in\mathscr{W}(\mathfrak{g})$ are linearly independant and we get $c(w^{-1},t)=c(w,s)$ for every $s\in\mathfrak{t}_0^{\mathrm{reg}}$. In particular, c does not depend on t and $c(w)=c(w^{-1})$. To prove Equation (6.3), we are going to prove that for every $\phi,\Psi\in\mathscr{C}_c^\infty(\mathrm{G}\,.\mathfrak{t}_0^{\mathrm{reg}})$, we have:

$$\int_{\mathfrak{t}_0\times\mathfrak{t}_0}\Pi(s)\hat{\mu}_t(s)M_{\mathfrak{t}_0}\phi(s)M_{\mathfrak{t}_0}\Psi(t)dsdt=\int_{\mathfrak{t}_0\times\mathfrak{t}_0}\Pi(t)\hat{\mu}_s(t)M_{\mathfrak{t}_0}\phi(s)M_{\mathfrak{t}_0}\Psi(t)dsdt.$$

We start with the left-hand side term.

$$\begin{aligned}&\int_{\mathfrak{t}_0\times\mathfrak{t}_0}\Pi(s)\hat{\mu}_t(s)M_{\mathfrak{t}_0}\phi(s)M_{\mathfrak{t}_0}\Psi(t)dsdt=\int_{\mathfrak{t}_0}\int_{\mathfrak{t}_0}\Pi(s)\hat{\mu}_t(s)\left(\Pi(s)\int_{\mathrm{G}}\phi(\mathrm{Ad}(g)s)dg\right)M_{\mathfrak{t}_0}\Psi(t)dsdt\\
=&\int_{\mathfrak{t}_0}\int_{\mathfrak{t}_0}\Pi(s)^2\left(\int_{\mathrm{G}}\hat{\mu}_t(\mathrm{Ad}(g)s)\phi(\mathrm{Ad}(g)s)dg\right)M_{\mathfrak{t}_0}\Psi(t)dsdt=(-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}\int_{\mathfrak{t}_0}\left(\int_{\mathfrak{g}_0}\hat{\mu}_t(X)\phi(X)dx\right)M_{\mathfrak{t}_0}\Psi(t)dt\\
=&(-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}\int_{\mathfrak{t}_0}\hat{\mu}_t(\phi)M_{\mathfrak{t}_0}\Psi(t)dt=(-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}\int_{\mathfrak{t}_0}\mu_t(\mathscr{F}_{\mathfrak{g}_0}\phi)M_{\mathfrak{t}_0}\Psi(t)dt\\
=&(-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}\int_{\mathfrak{t}_0}\Pi(t)\int_{\mathrm{G}}F_{\mathfrak{g}_0}\phi(\mathrm{Ad}(g)t)M_{\mathfrak{t}_0}\Psi(t)dgdt\\
=&(-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}\int_{\mathfrak{t}_0}\int_{\mathrm{G}}\int_{\mathrm{G}}\Pi(t)^2F_{\mathfrak{g}_0}\phi(\mathrm{Ad}(g)t)\Psi(\mathrm{Ad}(h)t)dhdgdt\\
=&\int_{\mathfrak{t}_0}\int_{\mathrm{G}}\int_{\mathrm{G}}|\Pi(t)|^2F_{\mathfrak{g}_0}\phi(\mathrm{Ad}(gh)t)\Psi(\mathrm{Ad}(h)t)dhdgdt=\int_{\mathrm{G}}\int_{\mathfrak{g}_0}F_{\mathfrak{g}_0}\phi(\mathrm{Ad}(g)X)\Psi(X)dXdg\\
=&\int_{\mathrm{G}}\int_{\mathfrak{g}_0}\int_{\mathfrak{g}_0}e^{i\mathscr{K}(\mathrm{Ad}(g)X,Y)}\phi(Y)\Psi(X)dYdXdg\end{aligned}$$

Same for the right-hand side. Which prove the first part.
For the second point, we already know that $c(w)=c(w^{-1})$. Wr prove the second part. We first observe that for every $v\in\mathscr{W}(\mathrm{T})$, we have $\mu_{v(t)}=\varepsilon(v)\mu_t$.
Indeed, v is anelement of $N_{\mathrm{G}}(\mathrm{T})/\mathrm{T}$, i.e. $v=\bar{g}=g\mathrm{T}$, $g\in N_{\mathrm{G}}(\mathrm{T})\subseteq\mathrm{G}$, then

$$\begin{aligned}\mu_{v(t)}\phi &= \Pi(v(t))\int_{\mathrm{G}}\phi(\mathrm{Ad}(g)(v(t)))dg=\varepsilon(v)\Pi(t)\int_{\mathrm{G}}\phi(\mathrm{Ad}(gv)(t))dg\\ &= \varepsilon(v)\Pi(t)\int_{\mathrm{G}}\phi(\mathrm{Ad}(gv)(t))dg=\varepsilon(v)\mu_t(\phi).\end{aligned}$$

By using Theorem 6.1.7, we get:

$$\Pi(s)\sum_{w\in\mathscr{W}(\mathfrak{g})}c(wv^{-1})e^{i\mathscr{K}(wt,s)}=\varepsilon(v)\Pi(s)\mathscr{W}(\mathfrak{g})c(w)e^{i\mathscr{K}(wt,s)},$$

which imply that $c(wv^{-1}) = \varepsilon(v)c(w)$. Which conclude the proof.

□

Remark 6.1.10. We directly get that

$$M_{\mathfrak{t}_0} F_{\mathfrak{g}_0}\phi(t) = (-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}} \sum_{w\in\mathscr{W}(\mathfrak{g})} c(w) F_{\mathfrak{t}_0} M_{\mathfrak{t}_0}\phi(wt)$$

for $t \in \mathfrak{t}_0^{\mathrm{reg}}$ and $\phi \in \mathscr{C}_c^\infty(\mathrm{G}\,.\mathfrak{t}_0^{\mathrm{reg}})$. Indeed,

$$\begin{aligned}
M_{\mathfrak{t}_0} F_{\mathfrak{g}_0}\phi(t) &= \Pi(t)\int_{\mathrm{G}} F_{\mathfrak{g}_0}\phi(\mathrm{Ad}(g)t)dg = \mu_t(F_{\mathfrak{g}_0}) = \hat{\mu}_t(\phi) \\
&= \int_{\mathfrak{g}_0} \hat{\mu}_t(X)\phi(X)dx = {}_{\mathfrak{t}_0}|\Pi(s)|^2 \int_{\mathrm{G}} \hat{\mu}_t(\mathrm{Ad}(g)s)\phi(\mathrm{Ad}(g)s)dgds \\
&= (-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}} \int_{\mathfrak{t}_0} \Pi(s)\hat{\mu}_t(s)\left(\int_{\mathrm{G}} \Pi(s)\phi(\mathrm{Ad}(g)s)dg\right)ds \\
&= (-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}} \int_{\mathfrak{t}_0} \Pi(s)\hat{\mu}_t(s)M_{\mathfrak{t}_0}\phi(s)ds \\
&= (-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}} \int_{\mathfrak{t}_0} \sum_{w\in\mathscr{W}(\mathfrak{g})} c(w)e^{i\mathscr{K}(wt,s)} M_{\mathfrak{t}_0}\phi(s)ds \\
&= (-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}} \sum_{w\in\mathscr{W}(\mathfrak{g})} c(w)\int_{\mathfrak{t}_0} e^{i\mathscr{K}(wt,s)} M_{\mathfrak{t}_0}\phi(s)ds \\
&= (-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}} \sum_{w\in\mathscr{W}(\mathfrak{g})} c(w)\int_{\mathfrak{t}_0} F_{\mathfrak{t}_0} M_{\mathfrak{t}_0}\phi(wt).
\end{aligned}$$

We admit the following for the moment.

Theorem 6.1.11 (Rossmann)**.** *We get:*

$$c(w) = \begin{cases} \dfrac{i^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}(-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}}{|\mathscr{W}(\mathrm{T})|}\varepsilon(w) & \text{if } w \in \mathscr{W}(\mathrm{T}) \\ 0 & \text{otherwise} \end{cases}.$$

The proof of this result can be found in [25, Equation 15].

Corollary 6.1.12. *The Fourier transform $\mu_t, t \in \mathfrak{t}_0^{\mathrm{reg}}$ is given on $\mathfrak{t}_0^{\mathrm{reg}}$ by the formula:*

$$\hat{\mu}_t(s) = K\Pi(s)^{-1} \sum_{w\in\mathscr{W}(\mathrm{T})} \varepsilon(w)e^{i\mathscr{K}(wt,s)}, \tag{6.4}$$

where $K = \frac{i^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}(-1)^{\frac{\dim_{\mathbb{C}}(\mathfrak{g}/\mathfrak{t})}{2}}}{|\mathscr{W}(\mathrm{T})|}$.

We finish this section with the following Lemma.

Lemma 6.1.13. *We denote by $\mathscr{W}(\mathfrak{k})$ the Weyl group corresponding to $(\mathfrak{k}, \mathfrak{t})$. Then, $\mathscr{W}(\mathfrak{k}) \approx \mathscr{W}(\mathrm{T})$.*

In order to prove this Lemma, we recall a standard result concerning Cartan subgroups.

Lemma 6.1.14. *Let $g = k\exp(p) \in \mathrm{G}$, where $k \in \mathrm{K}$, $p \in {}_0$ and $\mathscr{S}_0$ be a θ-stable of $\mathfrak{g}_0$ (see CITE) such that* $\mathrm{Ad}(g)$ *normalizes (resp. centralizes) $\mathfrak{s}_0$. Then,* $\mathrm{Ad}(k)$ *and* $\mathrm{ad}(p)$ *each normalizes (resp. centralizes) $\mathfrak{s}_0$.*

Proof of Lemma 6.1.13. From Lemma 6.1.14, we get that $N_{\mathrm{G}}(\mathfrak{t}_0) = N_{\mathrm{K}}(\mathfrak{t}_0)$. Indeed, the inclusion of $N_{\mathrm{K}}(\mathfrak{t}_0) \in N_{\mathrm{G}}(\mathfrak{t}_0)$ is obvious. Conversly, let $g \in N_{\mathrm{G}}(\mathfrak{t}_0)$ and $g = k\exp(p), k \in \mathrm{K}, p \in \mathfrak{p}_0$ the Iwasawa decomposition. Using Lemma 6.1.14, $k \in \mathrm{N}_{\mathrm{K}}(\mathfrak{t}_0)$ and $\mathrm{ad}(p)(\mathfrak{t}_0) \subseteq \mathfrak{t}_0$. Using $\mathfrak{t}_0 \subseteq \mathfrak{k}_0$ and $[\mathfrak{k}_0, \mathfrak{p}_0] \subseteq \mathfrak{p}_0$ (see Section 3.1). Then, $\mathrm{ad}(p)(\mathfrak{t}_0) \subseteq \mathfrak{t}_0 \cap \mathfrak{p}_0 = \{0\}$.
In particular, $g = k$ and then, $g \in \mathrm{N}_{\mathrm{K}}(\mathfrak{t}_0$. The result follows using Lemma 6.1.2.

□

6.2 A conjecture of Kirillov

In this section, we will exceptionally work with a general Lie group (it will not be useful for us but the theory is not more complicated in this context)
Let G be a real Lie group, $\mathfrak{g}_0$ and $\mathfrak{g}$ the Lie algebra of G and its complexification respectively and let $\mathfrak{g}_0^*$ and $\mathfrak{g}^*$ the dual spaces. We denote by Ad^* the action of G on $\mathfrak{g}_0$ (or $\mathfrak{g}$) given by

$$\mathrm{Ad}(g)\lambda(X) = \left\langle \lambda, \mathrm{Ad}(g^{-1})X \right\rangle \qquad (g \in \mathrm{G}, X \in \mathfrak{g}_0, \lambda \in \mathfrak{g}_0^*).$$

The differential of this action, denoted by $\mathrm{ad}^* : \mathfrak{g}_0 \to \mathrm{End}(\mathfrak{g}_0)$, is given by:

$$\mathrm{ad}^*(X)\lambda(Y) = -\langle \lambda, \mathrm{ad}(X)Y \rangle \qquad (X, Y \in \mathfrak{g}_0, \lambda \in \mathfrak{g}_0^*).$$

Fix $\lambda \in \mathfrak{g}_0^*$ and denote by $\Omega_\lambda = \mathrm{Ad}(\mathrm{G})\lambda$ be the corresponding co-adjoint orbit.

6.2.1 Liouville measure on a co-adjoint orbit

The first step is to construct a symplectic 2-form on Ω_λ

Lemma 6.2.1. *Let $\mu \in \Omega_\lambda$. Then, the tangent space $T_\mu\Omega_\lambda$ is given by:*

$$T_\mu\Omega_\lambda = \{\mathrm{ad}^*(\xi)\mu, \xi \in \mathfrak{g}_0\}.$$

Idea 1. For $\xi \in \mathfrak{g}_0$, we consider the map:

$$\mu_\xi : \mathbb{R} \ni t \to \mu_\xi(t) = \mathrm{Ad}(\exp(tX))\mu \in \Omega_\lambda.$$

It's a curve in Ω_λ such that $\mu_\xi(0) = \mu$. For $\eta \in \mathfrak{g}_0$, we get:

$$\left\langle \mu_\xi(t), \eta \right\rangle = \langle \mathrm{Ad}^*(\exp(t\xi))\mu, \eta \rangle = \langle \mu, \mathrm{Ad}(\exp(t\xi))\eta \rangle .$$

By differentiating with respect to t and taking $t = 0$, we get:

$$\left\langle \mu'_\xi(0), \eta \right\rangle = -\langle \mu, \mathrm{ad}(\xi)\eta \rangle = -\langle \mathrm{ad}^*(\xi)\mu, \eta \rangle ,$$

which implies that $\mu'_\xi(0) = -\mathrm{ad}^*(\xi)\mu$.

On $T_\mu\Omega_\lambda$, we define the form ω_μ by:

$$\omega_\mu(\mathrm{ad}^*(\xi)\mu, \mathrm{ad}^*(\eta)\mu) = \langle \mu, [\xi, \eta] \rangle . \tag{6.5}$$

Lemma 6.2.2. *The form ω_μ is a well-defined symplectic form on $T_\mu\Omega_\lambda$.*

Proof. Assume that $\omega_\mu(\mathrm{ad}^*(\xi)\mu, \mathrm{ad}^*(\eta)\mu) = 0$ for every $\mathrm{ad}^*(\eta)\mu$. Then, $\langle \mu, [\xi, \eta] \rangle = 0$ for every $\eta \in \mathfrak{g}_0$. Using that $\langle \mu, [\xi, \eta] \rangle = \langle \mathrm{ad}^*(\xi)\mu, \eta \rangle$, and then $\mathrm{ad}^*(\xi)\mu = 0$. In particular, ω_μ is non-degenerate.

□

Remark 6.2.3. We denote by $\mathfrak{g}_\mu$ the subalgebra of $\mathfrak{g}_0$ given by:

$$\mathfrak{g}_\mu = \{\xi \in \mathfrak{g}_0, \mathrm{ad}^*(\xi)\mu = 0\} ,$$

and by $\mathfrak{g}_0^\mu$ the subspace of $\mathfrak{g}_0^*$ given by:

$$\mathfrak{g}_0^\mu = \left\{ \nu \in \mathfrak{g}_0^*, \langle \nu, \eta \rangle = 0, \eta \in \mathfrak{g}_\mu \right\} .$$

Then, $T_\mu\Omega_\lambda = \mathfrak{g}_0^\mu$. Indeed, $T_\mu\Omega_\lambda \subseteq \mathfrak{g}_0^\mu$ because for every $\xi \in \mathfrak{g}_0$ and $\eta \in \mathfrak{g}_\mu$,

$$\langle \mathrm{ad}^*(\xi)\mu, \eta \rangle = \langle \mu, \mathrm{ad}(\xi)\eta \rangle = -\langle \mu, \mathrm{ad}(\eta)\xi \rangle = -\langle \mathrm{ad}^*(\eta)\mu, \xi \rangle = 0,$$

and we get the Equality using that $\dim(T_\mu\Omega_\lambda = \dim(\mathfrak{g}_0) - \dim(\mathfrak{g}_\mu) = \dim(\mathfrak{g}_0^\mu)$.

Using the form ω_μ on $T_\mu\Omega_\lambda, \mu \in \Omega_\lambda$, we define a form ω on Ω_λ in a natural way.

Exercice 28. Prove that the form ω is closed and G-invariant.

Corollary 6.2.4. *Co-adjoint orbits of finite dimensional Lie algebras are even dimensional.*

Let $2d$ be the dimension of Θ_λ. We denote by Θ_λ the form:

$$\Omega_\lambda = \frac{\omega^d}{(2\pi)^d d!}. \tag{6.6}$$

One can check [24] for more details. The form θ_λ is a volume form on Ω_λ. This form defines a tempered positive meqsure on $\mathfrak{g}_0$ denoted by $d\theta_\lambda$.

Remark 6.2.5. Assume that $\mathfrak{g}_0$ is semi-simple. In particular, the map:

$$\Psi : \mathfrak{g}_0 \ni X \to \Psi(X) = \mathscr{K}(X, \cdot) \in \mathfrak{g}_0^*,$$

is a. G-invariant bijective map. For $g \in \mathrm{G}$ and $X, Y \in \mathfrak{g}_0$, we get:

$$\begin{aligned}\Psi(\mathrm{Ad}(g)X)(Y) &= \mathscr{K}(\mathrm{Ad}(g)X, Y) = \mathscr{K}(X, \mathrm{Ad}^{-1}(g)Y) \\ &= \Psi(X)(\mathrm{Ad}(g^{-1})Y) = (\mathrm{Ad}^*(g)\Psi(X))(Y)\end{aligned}$$

In particular, $\Psi(\Omega_X = \mathrm{Ad}(\mathrm{G})(X)) = \Omega_{\Psi(X)}$. In particular, we have a one-to-one correspondence between $\mathfrak{g}_0/\,\mathrm{G}$ and $\mathfrak{g}_0^*/\,\mathrm{G}$.

The measure $d\theta_\lambda$ corresponding to the Liouville form θ_λ on Ω_λ defines a tempered distribution on $\mathfrak{g}_0$ given by:

$$d\theta_\lambda(\phi) = \int_{\Omega_\lambda} \phi(F) d\theta_\lambda(F) \qquad (\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0^*)).$$

Let $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0)$. Then, $\hat{\phi} = F_{\mathfrak{g}_0^*}\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0^*)$ and we can define the Fourier transform $d\hat{\theta}_\lambda$ of $d\theta_\lambda$ by:

$$d\hat{\theta}_\lambda(\phi) = d\theta_\lambda(F_{\mathfrak{g}_0^*}\phi) = \int_{\Omega_\lambda} F_{\mathfrak{g}_0^*}\phi(F) d\theta_\lambda(F) = \int_{\Omega_\lambda}\int_{\mathfrak{g}_0} \phi(X) e^{i\langle F, X\rangle} dX d\theta_\lambda(F),$$

where dX is the Lebesgue measure on $\mathfrak{g}_0$, i.e.e as a distribution:

$$d\hat{\theta}_\lambda(X) = \int_{\Omega_\lambda} e^{i\langle F, X\rangle} d\theta_\lambda(F) \qquad (X \in \mathfrak{g}_0).$$

Remark 6.2.6. The distribution $d\theta_\lambda$ is G-invariant. Indeed, for every $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0^*)$,

$$d\theta(\phi \circ \mathrm{Ad}^*(g^{-1})) = \int_{\Omega_\lambda} \phi \circ \mathrm{Ad}^*(g^{-1})(F) d\theta_\lambda(F) = \int_{\Omega_\lambda} \phi(F) d\theta_\lambda(F) = d\theta_\lambda(\phi),$$

because the measure $d\theta_\lambda$ is G-invariant (see Exercice 28).

6.2.2 How to understand Kirillov's conjecture?

Let U be an open neighbourhood of $\mathfrak{g}_0$ such that $\exp_{|_U}$ is injective. We denote by $V = \exp(U)$. In particular, we get a map:

$$\tau : \mathscr{C}_c^\infty(U) \ni \phi \to \gamma_\phi \in \mathscr{C}_c^\infty(V)$$

such that $\gamma_\phi(\exp(X)) = \phi(X), X \in U$. By dualizing the map τ, we get a map:

$$\tau^* : \mathscr{D}'(V) \ni T \to \tau * (T) \in \mathscr{D}'(U),$$

where $\tau^*(T)\phi = T(\gamma_\phi)$, where $\phi \in \mathscr{C}_c^\infty(U)$. Then, for a distribution on G given by a locally integrable function F_T on G, we get for every $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0)$ such that $\text{supp}(\phi) \subseteq U$,

$$\tau^*(T)\phi = T(\gamma_\phi) = \int_{\mathrm{G}} \gamma_\phi(g)F_T(g)dg = \int_{\mathfrak{g}_0} \gamma_\phi(\exp(X))F_T(\exp(X))p(X)dX$$

where $p(X) = \det\left(\frac{\sinh(\text{ad}(X/2))}{\text{ad}(X/2)}\right)^{\frac{1}{2}}$.

Conjecture 6.2.7 (Kirillov's Conjecture)**.** Let $(\Pi, \mathscr{H})$ be an irreducible unitary representation of G. Then, there exists $\lambda \in \mathfrak{g}_0^*$ such that:

$$\tau^*(\Theta_\Pi)(\phi) = d\hat{\theta}_\lambda(\phi),$$

where $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0)$ such that $\text{supp}(\phi) \subseteq U$, i.e. in terms of distributions,

$$\Theta_\Pi(\exp(X)) = p(X)^{-1} \int_{\Omega_\lambda} e^{i\langle F,X\rangle} d\theta_\lambda(F).$$

Remark 6.2.8. We have proved that for a semi-simple group, every irreducible quasi-simple representation $(\Pi, \mathscr{H})$ has a character (see 4) given by a locally integrable function on G analytic on G′ (see 5). In this context, using Remark 6.2.5, the previous conjecture can be stated as follow:

$$\int_{\mathfrak{g}_0} \phi(X)\Theta_\Pi(\exp(X))p(X)dX = \int_{\mathfrak{g}_0} \left(p(X)^{-1} \int_{\omega_Z} e^{i\langle Y,X\rangle} d\theta_Z(Y)\right)\phi(X)dX, \qquad (6.7)$$

where $Z \in \mathfrak{g}_0$ and $\phi \in \mathscr{C}_c^\infty(\mathfrak{g}_0$ such that $\text{supp}(\phi) \subseteq U$.

Exercice 29. Verify that Kirillov's conjecture holds for compact Lie groups using Equation 2.6.

6.3 Few words about discrete series representations

In this section, our reference is [21, Chapter 9-12]. We start with a proposition/definition.

Proposition 6.3.1. *For an irreducible unitary representation* $(\Pi, \mathscr{H})$ *of* G*, the following three conditions are equivalent:*

1. *Some non-zero* K*-finite matrix coefficient is in* $\mathrm{L}^2(\mathrm{G}, dg)$*,*

2. *All matrix coefficients are in* $\mathrm{L}^2(\mathrm{G})$*,*

3. Π *is equivalent with a direct summand of the right regular representation of* G *on* $\mathrm{L}^2(\mathrm{G}, dg)$.

Proof. See [21, Proposition 9.6].

□

Remark 6.3.2. When these confitions are satisfied, there exists a positive number d_Π such that:

$$\int_{\mathrm{G}} \langle \Pi(g)u_1, v_1\rangle \overline{\langle \Pi(g)u_2, v_2\rangle} dg = \frac{\langle u_1, u_2\rangle \overline{\langle v_1, v_2\rangle}}{d_\Pi}.$$

Notation 6.3.3. When these conditions are satisfied, we say that Π is in the discrete series of G and we call d_Π the formal degree of Π.

We recall some deep results of Harish-Chandra.

Theorem 6.3.4. *1. The group* G *has discrete series if and only if* $\mathrm{rk}(\mathrm{K}) = \mathrm{rk}(\mathrm{G})$.

2. *Let* T *be a maximal torus of* K *(we have assumed that* $\mathrm{rk}(\mathrm{K}) = \mathrm{rk}(\mathrm{G})$*). Suppose* $\lambda \in (i\mathfrak{t}_0)^*$ *is non singular with respect to* $\Phi(\mathfrak{g}, \mathfrak{t})$*, i.e.* $\langle \lambda, \alpha\rangle \neq 0$ *for every* $\alpha \in \Phi(\mathfrak{g}, \mathfrak{t})$ *and such that* $\langle \lambda, \alpha\rangle > 0$ *for every* $\alpha \in \Phi^+(\mathfrak{g}, \mathfrak{t})$.

   If $\lambda + \delta_{\mathrm{G}}$ *is analytically integral, then there exists a discrete series representation* Π_λ *of* G *with the following properties:*

   (a) Π_λ *has infinitesimal character* χ_λ,

   (b) $\Pi_{\lambda|_{\mathrm{K}}}$ *contains with multiplicity one the* K*-type with highest weight* $\Lambda = \lambda + \rho_{\mathrm{G}} - 2\rho_{\mathrm{K}}$,

   (c) If Λ' *is the highest weight of a* K*-type in* $\Pi_{\lambda|_{\mathrm{K}}}$*, then* Λ' *is of the form*

   $$\Lambda' = \Lambda + \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} n_\alpha \alpha \qquad (n_\alpha \in \mathbb{Z}_+).$$

   where $\rho_{\mathrm{G}} = \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} \alpha$ *and* $\rho_{\mathrm{K}} = \sum_{\alpha \in \Phi^+(\mathfrak{k},\mathfrak{t})} \alpha$*. Moreover, two representations* Π_λ *and* $\Pi_{\lambda'}$ *are equivalent if. and only if* λ *and* λ' *are conjugated under* $\mathscr{W}(T)$.

For the proof of this result, see [13, Theorem 13] and [21, Theorem 9.20].

Notation 6.3.5. The linear form λ of Theorem 6.3.4 is called the Harish-Chandra parameter and the K-type Λ is called the Blattner parameter.

The last theorem we are going to recall gives a precise formula for the character of the representation considered before.

Theorem 6.3.6. *1. We denote by* Θ_{Π_λ} *the character of the representation* Π_λ *given in Theorem 6.3.4. Then,*

$$\Theta_{\Pi_\lambda}(\exp(X)) = (-1)^{\frac{\dim(\mathrm{G}/\mathrm{K})}{2}} \sum_{w \in \mathscr{W}(\mathfrak{t})} \varepsilon(w) e^{w\lambda(X)} D(X) \qquad (X \in \mathfrak{t}_0^{\mathrm{reg}}).$$

2. *The only discrete series, up to equivalence, are the representations* Π_λ *of Theorem 6.3.4.*

For the proof of this theorem, see [13, Paragraph 40] and [21, Chapter XII].
We fix $t \in \mathfrak{t}_0^{\mathrm{reg}}$ and $\Omega_t = \mathrm{Ad}^*(\mathrm{G})(t)$ the corresponding orbit. Using Section 6.1, we define a measure ν_t on Ω_t by:

$$\int_{\Omega_t} f(X)d\nu_t(X) = \int_{\mathrm{G}} f(\mathrm{Ad}(g)t)dg \qquad (f \in \mathscr{C}_c^\infty(\mathfrak{g}_0)).$$

Exercice 30. 1. Prove that $d\theta_t = |\mathscr{W}(\mathrm{T})||\Pi(t)|\nu_t$

2. Prove Kirillov's conjecture for discrete series.

6.4 Enright's result for irreducible unitary highest weight modules

For more details about this result, see [1]. Let G, K, and T as before. Let be a Borel subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$ and $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{u}$ the corresponding maximal parabolic. We denote by $\mathfrak{u}$ the abelian nilradical of $\mathfrak{q}$, i.e. $\mathfrak{q} = \mathfrak{k} \oplus \mathfrak{u}$, by $\Phi(\mathfrak{u}, \mathfrak{t}) = \{\alpha \in \Phi(\mathfrak{g}, \mathfrak{t}), \mathfrak{g}_\alpha \subseteq \mathfrak{u}\}$ and by $\bar{\mathfrak{u}}$ the subspace $\sum_{\alpha \in \Phi(\mathfrak{u},\mathfrak{t})} \mathfrak{g}_{-\alpha}$.
In particular, $\mathfrak{g} = \bar{\mathfrak{u}} \oplus \mathfrak{q}$.
We denote by $\mathscr{W}^{\mathfrak{k}}$ the subset of $\mathscr{W}(\mathfrak{g})$ given by:

$$\mathscr{W}^{\mathfrak{k}} = \{w \in \mathscr{W}(\mathfrak{g}), w(\rho) \text{ is } \Phi^+(\mathfrak{g}, \mathfrak{t}) - \text{ dominant }\}.$$

Lemma 6.4.1. *We get* $\mathscr{W} = \mathscr{W}(\mathfrak{k}).\mathscr{W}^{\mathfrak{k}}$, *with* $\mathscr{W}(\mathfrak{k}) \cap \mathscr{W}^{\mathfrak{k}} = \{\mathrm{Id}\}$.

We denote by

$$\mathscr{W} \ni w \to \bar{w} \in \mathscr{W}^{\mathfrak{k}}$$

the corresponding projection.

Example 6.4.2. Let $\mathrm{G} = \mathrm{U}(p, q, \mathbb{C})$, $\mathrm{K} = \mathrm{U}(p, \mathbb{C}) \times \mathrm{U}(q, \mathbb{C})$, $\mathrm{T} \approx \mathrm{S}^{1^{p+q}}$. We use the notations of Section 3.1. We have:

$$\Phi(\mathfrak{g}, \mathfrak{t}) = \left\{\pm(e_i - e_j), 1 \leq i < j \leq p + q\right\},$$

$$\Phi(\mathfrak{k}, \mathfrak{t}) = \left\{\pm(e_i - e_j), 1 \leq i < j \leq p\right\} \cup \left\{\pm(e_i - e_j), p + 1 \leq i < j \leq p + q\right\}.$$

Moreover, $\mathscr{W}(\mathfrak{g}) \approx \mathscr{S}_{p+q}$, $\mathscr{W}(\mathfrak{k}) \approx \mathscr{S}_p \times \mathscr{S}_q$ and

$$\rho_{\mathrm{G}} = \frac{1}{2} \sum_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t})} \alpha = \frac{1}{2} \sum_{1 \leq i < j \leq p+q} e_1 - e_j = \sum_{i=1}^{p+q} \frac{p + q - 2i + 1}{2} e_i.$$

Similarly,

$$\rho_{\mathrm{K}} = \frac{1}{2}\sum_{\alpha\in\Phi^+(\mathfrak{g},\mathfrak{t})}\alpha = \sum_{i=1}^{p}\frac{p-2i+1}{2}e_i + \sum_{i=1}^{q}\frac{q-2i+1}{2}e_{p+i}.$$

We have $\Phi(\mathfrak{u},\mathfrak{t}) = \left\{e_i - e_j, 1\le i\le p, p+1\le j\le p+q\right\}$. From now one, we fix $p=q=2$. We get:

$$\mathscr{W}(\mathfrak{k}) = \{\mathrm{Id},(12),(34),(12)(34)\} \qquad \mathscr{W}^{\mathfrak{k}} = \{\mathrm{Id},(23),(132),(234),(13)(24),(1342)\}.$$

Finally, let $(123)\in\mathscr{W}$. Then, $(123) = (12)(23)$ and then $\overline{(123)} = (23)$.

Definition 6.4.3. Let $\lambda\in\mathscr{T}^*$. We define $\mathscr{W}_\lambda$ to be the subgroup of $\mathscr{W}(\mathfrak{g})$ generated by the identity element and the reflections s_α which satisfy:

1. $\alpha\in\Phi(\mathfrak{u},\mathfrak{t})$ and $\langle\lambda,\widetilde{\alpha}\rangle\in\mathbb{Z}_+^*$, where $\widetilde{\alpha} = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$,
2. If $\beta\in\Phi(\mathfrak{g},\mathfrak{t})$ and $\langle\lambda,\beta\rangle = 0$, then $\langle\alpha,\beta\rangle = 0$,
3. If $\beta\in\Phi(\mathfrak{g},\mathfrak{t})$ is long and $\langle\lambda,\beta\rangle = 0$, then α is short.

We denote by Φ_λ the set of $\alpha\in\Phi(\mathfrak{g},\mathfrak{t})$ satisfying the previous conditions. We denote by $\mathscr{W}_\lambda(\mathfrak{k})$ the subset of $\mathscr{W}(\mathfrak{k})$ defined by

$$\mathscr{W}_\lambda(\mathfrak{k}) = \mathscr{W}_\lambda\cap\mathscr{W}(\mathfrak{k}).$$

and by $\mathscr{W}_\lambda^{\mathfrak{k}}$ the following set:

$$\mathscr{W}_\lambda^{\mathfrak{k}} = \bigcup_{i\in\mathbb{Z}_+^*}\{w\in\mathscr{W}_\lambda.w\rho \text{ is } \Phi_\lambda^+(\mathfrak{k})-\text{ dominant }, l_\lambda(w) = i\},$$

where $\Phi_\lambda^+(\mathfrak{k}) = \Phi_\lambda\cap\Phi^+(\mathfrak{k},\mathfrak{t})$ and l_λ is the length of w determined by $\Phi_\lambda^+(\mathfrak{k})$.

Remark 6.4.4. We have the following decomposition $\mathscr{W}_\lambda = \mathscr{W}_\lambda(\mathfrak{k})\times\mathscr{W}_\lambda^{\mathfrak{k}}$.

Example 6.4.5.

1. Let $\lambda = \sum_{a=1}^{p+q}\lambda_a e_a, \lambda_a\in\mathbb{Z}_+^*,\ \lambda_i > \lambda_j, i > j$. For every $a\in[|1,p|], b\in[|p+1,p+q|]$, we have:

   $$\langle\lambda, e_a - e_b\rangle = e_a - e_b > 0.$$

   Moreover, for every $\beta\in\Phi(\mathfrak{g},\mathfrak{t})$, $\langle\lambda,\beta\rangle\neq 0$. Then,

   $$\mathscr{W}_\lambda = \{s_\alpha, \alpha\in\Phi^+(\mathfrak{g},\mathfrak{t})\ \Phi(\mathfrak{k},\mathfrak{t})\}.$$

   In particular, $\mathscr{W}_\lambda = \mathscr{W}$. Indeed, fix $1\le a\le p, 1\le b\le p$, then, for every $p+1\le c\le p+q$, we get $(ab) = (ac)(bc)(ac)$.

2. Fix $p = q = 2$ and $\lambda = 2\rho_{\mathrm{K}} = e_1 - e_2 + e_3 - e_4$. We have:

   $$\langle\lambda, e_1 - e_3\rangle = \langle\lambda, e_2 - e_4\rangle = 0.$$

We have $\langle \lambda, e_1 - e_4 \rangle = 2$ and $\langle e_1 - e_4, e_1 - e_3 \rangle = 1$. Then, $s_{e_1 - e_4} \notin \mathscr{W}_\lambda$. Finally, $\mathscr{W}_\lambda = \{\mathrm{Id}\}$.

Theorem 6.4.6. *Let* $(\Pi, \mathscr{H})$ *be an irreducible unitary representation of highest weight* $\lambda - \rho$. *Then,*

$$\prod_{\alpha \in \Phi^+(\mathfrak{g},\mathfrak{t}) \setminus \Phi^+(\mathfrak{k},\mathfrak{t})} \left(e^{\frac{\alpha(X)}{2}} - e^{\frac{\alpha(X)}{2}}\right) \Theta_\Pi(\exp(X)) = \sum_{w \in \mathscr{W}_\lambda^{\mathfrak{k}}} (-1)^{l_{\lambda(w)}} \Theta(\mathrm{K}, \bar{w}\lambda - \rho_{\mathrm{K}})(\exp(X)),$$

where $\Theta(\mathrm{K}, \bar{w}\lambda - \rho_{\mathrm{K}})$ *is the character of an irreducible representation of* $|K$ *of highest weight* $\bar{w}\lambda - \rho_{\mathrm{K}}$ *and* $X \in \mathfrak{t}_0^{\mathrm{reg}}$.

6.5 Open Problems

6.5.1 Enright's result and Kirillov's conjecture

In Section 6.4, we recalled a result of Enright concerning the character of an irreducible unitary highest weight module. This formula gives a formula for

$$p(X)\Theta_\Pi(\exp(X)) \qquad (X \in \mathfrak{t}_0^{\mathrm{reg}}).$$

In Section 6.2, we recalled a conjecture of Kirillov (see Equation 6.7).

Questions 6.5.1. Using Equation (6.4), can we prove Kirillov formula for unitary highest weight modules?

6.5.2 Howe duality and Kirillov's conjecture

Let $(W, \langle \cdot, \cdot \rangle)$ be a real symplectic space, $\mathrm{Sp}(W)$ be the corresponding group of isometries and $(\omega, \mathscr{H})$ be the metaplectic representation of the connected double cover $\widetilde{\mathrm{Sp}(W)}$ of the symplectic group. A dual pair in $\mathrm{Sp}(W)$ is a pair of subgroups (G, G') of $\mathrm{Sp}(W)$ such that the centralizer of G (resp. G') in $\mathrm{Sp}(W)$ is G (resp. G'). The dual pair is said irreducible if we cannot find an orthogonal decomposition of W of the form $W = W_1 \oplus W_2$, where both W_1 and W_2 are $\mathrm{G} \cdot \mathrm{G}'$-invariant, and is said reductive if the natural action of G and G' on W is completely reductive. The irreducible reductive dual pairs had been classified (see [17]).

In [19], R. Howe proved that we have a one-to-one correspondence between $\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ and $\mathscr{R}(\widetilde{\mathrm{G}'}, \omega)$ whose graph is $\mathscr{R}(\widetilde{\mathrm{G}}.\cdot\widetilde{\mathrm{G}'}, \omega)$, where for any subgroup $\widetilde{\mathrm{H}}$ of $\widetilde{\mathrm{Sp}(W)}$ the set $\mathscr{R}(\widetilde{\mathrm{H}}, \omega)$ corresponds to the set of infinitesimal equivalence classes of irreducible admissible representations of $\widetilde{\mathrm{H}}$ which can be realised as a quotient of $\mathscr{H}^\infty$ by a closed $\omega^\infty(\widetilde{\mathrm{H}})$-invariant subspace. When G is compact, the duality can be stated in an easier way: we get the following decomposition:

$$\omega_{|_{\widetilde{\mathrm{G}} \cdot \widetilde{\mathrm{G}'}}} = \bigoplus_{(\Pi, V_\Pi) \in \widetilde{\mathrm{G}}_\omega} \Pi \otimes \Pi'$$

where $\widetilde{G}_\omega$ is the set of irreducible unitary representations of $\widetilde{G}$ such that $\mathrm{Hom}_{\widetilde{G}}(V_\Pi, \mathscr{H}) \neq \{0\}$ and Π' is an irreducible unitary representation of $\widetilde{G}'$.

Remark 6.5.2. Assume that G is compact. We fix a representation $(\Pi, V_\Pi) \in \widetilde{G}_\omega$ and denote by $\mathscr{P}_\Pi : \mathscr{H} \to \mathscr{H}(\Pi)$ the projection onto the Π-isotypic component. For every $\Psi \in \mathscr{C}_c^\infty(G')$, we get:

$$\Theta_{\Pi'}(\Psi) = \mathrm{tr}(\mathscr{P}_\Pi \omega(\Psi) = \mathrm{tr} \int_{\widetilde{G}'} \left(\int_{\widetilde{G}} \overline{\Theta_\Pi(\widetilde{g})} \omega(\widetilde{g}\widetilde{g}') d\widetilde{g} \right) d\widetilde{g}' = \int_{\widetilde{G}'} \left(\int_{\widetilde{G}} \overline{\Theta_\Pi(\widetilde{g})} \Theta(\widetilde{g}\widetilde{g}') d\widetilde{g} \right) d\widetilde{g}',$$

where Θ is the character of ω. Then, by using the oscillator semigroup introduced by Howe in [18], we get that the character $\Theta_{\Pi'}$ on G'^{reg} is given by:

$$\Theta_{\Pi'}(\widetilde{g'}) = \lim_{\substack{\widetilde{p} \to 1 \\ \widetilde{p} \in \widetilde{G'^{++}}}} \int_{\widetilde{G}} \overline{\Theta_\Pi(\widetilde{g})} \Theta(\widetilde{g}\widetilde{g'}\widetilde{p}) d\widetilde{g} \qquad (g' \in G'^{++}),$$

where $G'^{++} = G'_{\mathbb{C}} \cap \mathrm{Sp}(W_{\mathbb{C}})^{++}$. We computed the value of this integral on the maximal compact torus T' of $G' = U(p, q, \mathbb{C})$. For the dual pair $(G = U(1, \mathbb{C}), G' = U(1, 1, \mathbb{C}))$, we are able to get a the value of $\Theta_{\Pi'}$ on the non-compact torus of G' (unique up to conjugation).

Questions 6.5.3. Can we prove that Kirillov conjecture is satisfied for Π' if we assume that Kirillov's conjecture is valid for Π?

Appendix A

Proof of Theorem 5.6.10

.

Bibliography

[1] Thomas J. Enright. Analogues of Kostant's u-cohomology formulas for unitary highest weight modules. *J. Reine Angew. Math.*, 392:27–36, 1988.

[2] Lars Gå rding. Note on continuous representations of Lie groups. *Proc. Nat. Acad. Sci. U.S.A.*, 33:331–332, 1947.

[3] Roe Goodman and Nolan R. Wallach. *Symmetry, representations, and invariants*, volume 255 of *Graduate Texts in Mathematics*. Springer, Dordrecht, 2009.

[4] Harish-Chandra. Representations of a semisimple Lie group on a Banach space. I. *Trans. Amer. Math. Soc.*, 75:185–243, 1953.

[5] Harish-Chandra. Representations of semisimple Lie groups. II. *Trans. Amer. Math. Soc.*, 76:26–65, 1954.

[6] Harish-Chandra. Representations of semisimple Lie groups. III. *Trans. Amer. Math. Soc.*, 76:234–253, 1954.

[7] Harish-Chandra. The characters of semisimple Lie groups. *Trans. Amer. Math. Soc.*, 83:98–163, 1956.

[8] Harish-Chandra. Fourier transforms on a semisimple Lie algebra. I. *Amer. J. Math.*, 79:193–257, 1957.

[9] Harish-Chandra. Invariant eigendistributions on semisimple Lie groups. *Bull. Amer. Math. Soc.*, 69:117–123, 1963.

[10] Harish-Chandra. Invariant distributions on Lie algebras. *Amer. J. Math.*, 86:271–309, 1964.

[11] Harish-Chandra. Discrete series for semisimple Lie groups. I. Construction of invariant eigendistributions. *Acta Math.*, 113:241–318, 1965.

[12] Harish-Chandra. Invariant eigendistributions on a semisimple Lie group. *Trans. Amer. Math. Soc.*, 119:457–508, 1965.

[13] Harish-Chandra. Discrete series for semisimple Lie groups. II. Explicit determination of the characters. *Acta Math.*, 116:1–111, 1966.

[14] Sigurđ ur Helgason. Differential operators on homogeneous spaces. *Acta Math.*, 102:239–299, 1959.

[15] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001. Corrected reprint of the 1978 original.

[16] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[17] Roger Howe. Preliminaries i. (unpublished).

[18] Roger Howe. The oscillator semigroup. In *The mathematical heritage of Hermann Weyl (Durham, NC, 1987)*, volume 48 of *Proc. Sympos. Pure Math.*, pages 61–132. Amer. Math. Soc., Providence, RI, 1988.

[19] Roger Howe. Transcending classical invariant theory. *J. Amer. Math. Soc.*, 2(3):535–552, 1989.

[20] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1978. Second printing, revised.

[21] Anthony W. Knapp. *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.

[22] Anthony W. Knapp. *Lie groups beyond an introduction*, volume 140 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, second edition, 2002.

[23] Anthony W. Knapp and David A. Vogan, Jr. *Cohomological induction and unitary representations*, volume 45 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1995.

[24] Paul-Emile Paradan. The Fourier transform of semi-simple coadjoint orbits. *J. Funct. Anal.*, 163(1):152–179, 1999.

[25] Wulf Rossmann. *Lie groups*, volume 5 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. An introduction through linear groups.

[26] V. S. Varadarajan. *Harmonic analysis on real reductive groups*. Lecture Notes in Mathematics, Vol. 576. Springer-Verlag, Berlin-New York, 1977.

[27] Nolan R. Wallach. *Real reductive groups. I*, volume 132 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1988.