CLASSIFICATION OF DUAL PAIRS IN THE SYMPLECTIC GROUP

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Let W be a finite dimensional vector space over $\mathbb{K}$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, endowed with a non-degenerate, skew-symmetric, bilinear form $\langle\cdot,\cdot\rangle$. We denote by Sp(W) the group of isometries corresponding to $(\mathrm{W}, \langle\cdot,\cdot\rangle)$, i.e.

$$\mathrm{Sp(W)} = \{g \in \mathrm{GL(W)}, \langle g(u), g(v)\rangle = \langle u, v\rangle, u, v \in \mathrm{W}\} ,$$

and let $\mathfrak{sp}(\mathrm{W})$ be the Lie algebra of Sp(W).

Definition 0.1. We say that a pair of subgroups $(\mathrm{G}, \mathrm{G}')$ of Sp(W) is a dual pair if G is the centralizer of G' in Sp(W) and vice-versa. The dual pair is said to be reductive if both G and G' act reductively on W.

Remark 0.2. Assume that there exists a $\langle\cdot,\cdot\rangle$-orthogonal decomposition of W of the form $\mathrm{W} = \mathrm{W}^1 \oplus^\perp \mathrm{W}^2$, where both W^1 and W^2 are G and G'-invariant. Then, $(\mathrm{G}_1, \mathrm{G}'_1)$ (resp. $(\mathrm{G}_2, \mathrm{G}'_2)$) is a dual pair in $\mathrm{Sp}(\mathrm{W}^1, \langle\cdot,\cdot\rangle_1)$ (resp. $\mathrm{Sp}(\mathrm{W}^2, \langle\cdot,\cdot\rangle_2)$), where $\mathrm{G}_i = \mathrm{G}_{|_{\mathrm{W}^i}}$, $\mathrm{G}'_i = \mathrm{G}'_{|_{\mathrm{W}^i}}$ and $\langle\cdot,\cdot\rangle_i = \langle\cdot,\cdot\rangle_{|_{\mathrm{W}^i\times\mathrm{W}^i}}$.
In this case, we say that $(\mathrm{G}, \mathrm{G}')$ is the direct sum of $(\mathrm{G}_1, \mathrm{G}'_1)$ and $(\mathrm{G}_2, \mathrm{G}'_2)$.

Definition 0.3. The dual pair is said to be irreducible if we cannot find a decomposition of W as in Remark 0.2.

Before giving a classification of the irreducible reductive dual pairs in the symplectic group, we are going to prove that every reductive dual pair is a direct sum of irreducible reductive dual pairs. We divide this proof in a couple of lemmas. Let $(\mathrm{G}, \mathrm{G}')$ be a reductive dual pair in Sp(W). Because the action of G on W is reductive, we have

$$\mathrm{W} = \bigoplus_{i=1}^{n} \mathrm{W}_{\lambda_i} = \bigoplus_{i=1}^{n} \left(\bigoplus_{j=1}^{d_i} \mathrm{T}_j(\mathrm{W}_i) \right) ,$$

where $(\mathrm{W}_i, \lambda_i)$ is an irreducible G-module, W_{λ_i} is the corresponding G-isotypic component in W, i.e.

$$\mathrm{W}_{\lambda_i} = \{\mathrm{T}(\mathrm{W}_i), \mathrm{T} \in \mathrm{Hom_G}(\mathrm{W}_i, \mathrm{W})\} ,$$

$d_i = \dim_{\mathbb{K}} \mathrm{Hom_G}(\mathrm{W}_i, \mathrm{V})$ and $\mathrm{T}_1, \ldots, \mathrm{T}_{d_i}$ is a $\mathbb{K}$-basis of $\mathrm{Hom_G}(\mathrm{W}_i, \mathrm{V})$.

Notation 0.4. For every $1 \leq i \leq n$ and $1 \leq j \leq d_i$, we denote by W_i^j the subspace of W_{λ_i} given by $\mathrm{W}_i^j = \mathrm{T}_j(\mathrm{W}_i)$.

Lemma 0.5. *The restriction of* $\langle\cdot,\cdot\rangle$ *to* W_i^j *is either zero or non-degenerate.*

Proof. The subspace of W_i^j given by

$$\left\{v \in \mathrm{W}_i^j, \langle v, w\rangle = 0, w \in \mathrm{W}_i^j\right\}$$

G-invariant. In particular, this space is either $\{0\}$ (i.e. $\langle\cdot,\cdot\rangle : \mathrm{W}_i^j \times \mathrm{W}_i^j \to \mathbb{K}$ is non-degenerate) or W_i^j (i.e. $\langle\cdot,\cdot\rangle : \mathrm{W}_i^j \times \mathrm{W}_i^j \to \mathbb{K}$ is zero).

□

Remark 0.6. (1) Assume that the restriction of $\langle \cdot, \cdot \rangle$ to W_i^j is non-degenerate. It follows that the G-module W_i^j is self-dual (the G-equivariant map $\mathrm{S} : \mathrm{W}_i^j \to \mathrm{W}_i^{j*}$ is given by $\mathrm{S}(v)(w) = \langle u, v \rangle, v, w \in \mathrm{W}_i^j$).

(2) Assume that the restriction of W_i^j is zero. Then, there exists $1 \leq k \leq n$ and $1 \leq l \leq d_k$ such that the form $\langle \cdot, \cdot \rangle$ on $\mathrm{W}_i^j \times \mathrm{W}_k^l$ is non-zero (otherwise, W_i^j will be in the radical of $\langle \cdot, \cdot \rangle$). In particular, by using that

$$\left\{ v \in \mathrm{W}_i^j, \langle v, w \rangle = 0, w \in \mathrm{W}_k^l \right\}$$

is G-invariant, it follows that the previous space is $\{0\}$, and we get a natural map

$$\mathrm{W}_i^j \to \mathrm{W}_k^{l\,*}$$

which is injective and G-invariant.

Lemma 0.7. *Let $i \in [|1, n|]$ such that the restriction of $\langle \cdot, \cdot \rangle$ to $\mathrm{W}_{\lambda_i} \times \mathrm{W}_{\lambda_i}$ is non-zero. Then, $\langle \cdot, \cdot \rangle$ is non-degenerate on W_{λ_i} and the space W_{λ_i} is $\langle \cdot, \cdot \rangle$-orthogonal to every $\mathrm{W}_{\lambda_j}, j \neq i$.*

Proof. Fix such i. Because the form $\langle \cdot, \cdot \rangle$ on $\mathrm{W}_{\lambda_i} \times \mathrm{W}_{\lambda_i}$ is non-zero, there exists $1 \leq k, t \leq d_i$ such that $\langle (\mathrm{W}_i^k, \mathrm{W}_i^t \rangle \neq \{0\}$. It follows from Lemma 0.5 that $\mathrm{W}_i^k \approx \mathrm{W}_i^{t*}$. By using that $\mathrm{W}_i^k \approx \mathrm{W}_i^t$, it follows that $\mathrm{W}_i^k \approx \mathrm{W}_i^{k*}$ as a G-module. In particular, $\mathrm{W}_i^a \approx \mathrm{W}_i^{a*}$ as a G-module for every $1 \leq a \leq d_i$.
Let $1 \leq j \leq n$ such that $i \neq j$, and $\mathrm{W}_j^b, 1 \leq b \leq d_j$, be an irreducible G-module of W_{λ_j}. If there exists $1 \leq a \leq d_i$ such that $\langle \mathrm{W}_i^a, \mathrm{W}_j^b \rangle \neq \{0\}$, it follows from Lemma 0.5 that $\mathrm{W}_i^a \approx \mathrm{W}_j^{b*}$. By using that $\mathrm{W}_i^a \approx \mathrm{W}_i^{a*}$, it follows that $\mathrm{W}_i^a \approx \mathrm{W}_j^b$, which is impossible. In particular, $\mathrm{W}_{\lambda_i} \perp \mathrm{W}_{\lambda_j}$, and it follows that $\mathrm{Rad}(\langle \cdot, \cdot \rangle_{\mathrm{W}_{\lambda_i} \times \mathrm{W}_{\lambda_i}}) \subseteq \mathrm{Rad}(\langle \cdot, \cdot \rangle) = \{0\}$.

□

Lemma 0.8. *Let $i \in [|1, n|]$ such that the restriction of $\langle \cdot, \cdot \rangle$ to $\mathrm{W}_{\lambda_i} \times \mathrm{W}_{\lambda_i}$ is zero. Then, there exists a unique $j \neq i$ such that $\langle \cdot, \cdot \rangle : \mathrm{W}_{\lambda_i} \oplus \mathrm{W}_{\lambda_j} \times \mathrm{W}_{\lambda_i} \oplus \mathrm{W}_{\lambda_j} \to \mathbb{K}$ is non-degenerate. Moreover, $\mathrm{W}_{\lambda_i} \oplus \mathrm{W}_{\lambda_j} \perp \mathrm{W}_{\lambda_k}, k \neq i, j$.*

By using the previous lemma, we get the following decomposition of W:

$$\mathrm{W} = (\mathrm{W}_{\lambda_1} \oplus \ldots \oplus \mathrm{W}_{\lambda_m}) \bigoplus ((\mathrm{W}_{\lambda_{m+1}} \oplus \mathrm{W}_{\lambda_{m+2}}) \oplus \ldots \oplus (\mathrm{W}_{\lambda_{n-1}} \oplus \mathrm{W}_{\lambda_n})),$$

where the restriction of $\langle \cdot, \cdot \rangle$ to $\mathrm{W}_{\lambda_1}, \ldots, \mathrm{W}_{\lambda_m}$ is non-degenerate, is zero on $\mathrm{W}_{\lambda_{m+1}}, \ldots, \mathrm{W}_{\lambda_n}$ such that the restriction of $\langle \cdot, \cdot \rangle$ to $\mathrm{W}_{\lambda_{m+1}} \oplus \mathrm{W}_{\lambda_{m+2}}, \ldots, \mathrm{W}_{\lambda_{n-1}} \oplus \mathrm{W}_{\lambda_n}$ is non-degenerate.

Notation 0.9. For every $1 \leq k \leq m + \frac{n-m}{2}$, we denote by $\widetilde{\mathrm{W}}^k$ the subspace of V given by

$$\widetilde{\mathrm{W}}^k = \begin{cases} \mathrm{W}_{\lambda_k} & \text{if } 1 \leq k \leq m \\ \mathrm{W}_{\lambda_{m+2(k-m)-1}} \oplus \mathrm{W}_{\lambda_{m+2(k-m)}} & \text{otherwise} \end{cases}$$

where the restriction of $\langle \cdot, \cdot \rangle$ to $\widetilde{\mathrm{W}}_{\lambda_k}, 1 \leq k \leq m$, is non-degenerate and $\widetilde{\mathrm{W}}^k, m+1 \leq k \leq m + \frac{n-m}{2}$ is a direct sum of two ($\langle \cdot, \cdot \rangle$-isotropic) G-isotypic component such that $\langle \cdot, \cdot \rangle_{|\widetilde{\mathrm{W}}^k \times \widetilde{\mathrm{W}}^k}$ is non-degenerate.

Corollary 0.10. *For every $1 \leq i \neq j \leq m + \frac{n-m}{2}$, the spaces $\widetilde{\mathrm{W}}^i$ and $\widetilde{\mathrm{W}}^j$ are $\langle \cdot, \cdot \rangle$-orthogonal.*

Proof. Obvious using Lemmas 0.7 and 0.8.

□

Proposition 0.11. *The joint action of* G *and* G′ *on* $\mathrm{W}^i, 1 \leq i \leq n$, *is irreducible.*

Proof. Let $\mathrm{U} \subseteq \mathrm{V}$ be an irreducible $\mathrm{G}\cdot\mathrm{G}'$-module. Because the form $\langle\cdot,\cdot\rangle$ is G and G′ invariant, it follows that the space

$$\{s \in \mathrm{U}, \langle s, u\rangle = 0, u \in \mathrm{U}\}$$

is either $\{0\}$ or U, i.e. $\langle\cdot,\cdot\rangle_{|\mathrm{U}\times\mathrm{U}}$ is either zero or non-degenerate.
Assume first that $\langle\cdot,\cdot\rangle_{|\mathrm{U}\times\mathrm{U}}$ is non-degenerate. Then, $\mathrm{W} = \mathrm{U} \oplus \mathrm{U}^{\perp}$, where $\mathrm{U}^{\perp}$ is given by:

$$\mathrm{U}^{\perp} = \{y \in \mathrm{W}, \langle y, u\rangle = 0, u \in \mathrm{U}\}\,.$$

Note that the space $\mathrm{U}^{\perp}$ is $\mathrm{G}\cdot\mathrm{G}'$-invariant and $\langle \mathrm{U}, \mathrm{U}^{\perp}\rangle = \{0\}$. Let $\mathrm{T} : \mathrm{W} \to \mathrm{W}$ be the map given by

$$\mathrm{T}(u + u^{\perp}) = u - u^{\perp}\,, \qquad (u \in \mathrm{U}, u^{\perp} \in \mathrm{U}^{\perp})\,.$$

One can easily check that $\mathrm{T} \in \mathrm{Sp}(\mathrm{W})$. Moreover, for every $g \in \mathrm{G}$, $u \in \mathrm{U}$ and $u^{\perp} \in \mathrm{U}^{\perp}$, we get:

$$\Pi(g) \circ \mathrm{T}(u + u^{\perp}) = \Pi(g)(u - u^{\perp}) = \Pi(g)(u) - \Pi(g)(u^{\perp}) = \mathrm{T} \circ \Pi(g)(u + u^{\perp}).$$

It implies that T commutes with G, i.e. $\mathrm{T} \in \mathrm{G}'$. Similarly, T commutes with G′, i.e. $\mathrm{T} \in \mathrm{G}$. Let (A, λ_0) and (B, λ_1) be two irreducible G-modules such that $\mathrm{A} \subseteq \mathrm{U}, \mathrm{B} \subseteq \mathrm{U}^{\perp}$. If A and B are equivalent, there exists a map $\mathrm{T}_{\mathrm{A},\mathrm{B}} : \mathrm{A} \to \mathrm{B}$ such that

$$\mathrm{T}_{\mathrm{A},\mathrm{B}} \circ \lambda_0(g) = \lambda_1(g) \circ \mathrm{T}_{\mathrm{A},\mathrm{B}}\,, \qquad (g \in \mathrm{G})\,.$$

In particular, for $g = \mathrm{T}$, we get for every $u \in \mathrm{U}$ that $\mathrm{T}_{\mathrm{A},\mathrm{B}}(u) = -\mathrm{T}_{\mathrm{A},\mathrm{B}}(u)$, which is impossible, i.e. that A and B are not isomorphic as G-modules. We get similar results for G′, and it follows that U is a full isotypic component for both G and G′.
Assume now that $\langle\cdot,\cdot\rangle_{|\mathrm{U}\times\mathrm{U}}$ is zero. As explained before, there exists an irreducible $\mathrm{G}\cdot\mathrm{G}'$-module $\mathrm{V} \subseteq \mathrm{W}$ such that the pairing $\langle\cdot,\cdot\rangle : \mathrm{U} \times \mathrm{V} \to \mathbb{K}$ is non-zero, and hence non-degenerate. One can easily see that the restriction of $\langle\cdot,\cdot\rangle$ to V is zero. As before, by using that $\langle\cdot,\cdot\rangle : \mathrm{U}\oplus\mathrm{V}\times\mathrm{U}\oplus\mathrm{V} \to \mathbb{K}$ is non-degenerate, we get that $\mathrm{W} = (\mathrm{U} \oplus \mathrm{V}) \oplus (\mathrm{U} \oplus \mathrm{V})^{\perp}$, and $(\mathrm{U} \oplus \mathrm{V})^{\perp}$ is $\mathrm{G}\cdot\mathrm{G}'$-invariant. Let $\mathrm{T}^t_1 : \mathrm{W} \to \mathrm{W}, t \in \mathbb{K}^*$, be the element of End(W) given by

$$\mathrm{T}^t_1(u) = tu, \qquad \mathrm{T}^t_1(v) = t^{-1}v, \qquad \mathrm{T}^t_1(a) = a, \qquad (u \in \mathrm{U}, v \in \mathrm{V}, a \in (\mathrm{U} \oplus \mathrm{V})^{\perp}).$$

One can easily see that for every $t \in \mathbb{K}^*$, $\mathrm{T}^t_1 \in \mathrm{Sp}(\mathrm{W})$, and by using the previous method, it follows that U is a full isotypic component for both G and G′. The proposition follows.

□

Proposition 0.12. *For every* $1 \leq k \leq m + \frac{n-m}{2}$, $(\mathrm{G}_{\widetilde{\mathrm{W}}^k}, \mathrm{G}'_{\widetilde{\mathrm{W}}^k})$ *is an irreducible dual pair in* $\mathrm{Sp}(\widetilde{\mathrm{W}}^k, \langle\cdot,\cdot\rangle_k)$.

Corollary 0.13. *Every reductive dual pair is a direct sum of irreducible reductive dual pairs.*

Definition 0.14. Let $(\mathrm{G}, \mathrm{G}')$ be an irreducible reductive dual pair in Sp(W). If the joint action of G and G′ on W is irreducible, we say that $(\mathrm{G}, \mathrm{G}')$ is of type I. Moreover, if $\mathrm{W} = \mathrm{W}^1 \oplus \mathrm{W}^2$ is a direct sum of two isotropic subspaces, where both W^1 and W^2 are $\mathrm{G}\cdot\mathrm{G}'$-invariant, we say that $(\mathrm{G}, \mathrm{G}')$ is of type II.

The goal is now to give a precise classification of reductive dual pairs in the (real or complex) symplectic group. According to Corollary 0.13, we can focus our intention on the irreducible ones.
From now on, we assume that $(\mathrm{G}, \mathrm{G}')$ is a reductive irreducible dual pair in Sp(W). We denote by Π the natural action of Sp(W) on W.

Lemma 0.15. *Assume that* $(\mathrm{G}, \mathrm{G}')$ *is of type I. Then, there exists a division algebra* $\mathbb{D}$ *over* $\mathbb{K}$*, a left vector space* V *over* $\mathbb{D}$*, a right vector space* V′ *over* $\mathbb{D}$ *such that*

$$\mathrm{W} = \mathrm{V}' \otimes_{\mathbb{D}} \mathrm{V}$$

and such that G *(resp.* G′*) acts on* V *(resp.* V′*) irreducibly (both actions are* $\mathbb{D}$*-linear).*

Proof. Let (λ, V) be an irreducible G-submodule of (Π, W), and let V_λ be the λ-isotypic component in W, i.e.

$$\mathrm{V}_\lambda = \{\mathrm{T}(\mathrm{V}), \mathrm{T} \in \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})\}\ .$$

One can easily see that for every $g' \in \mathrm{G}'$ and $\mathrm{T} \in \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})$, $\Pi(g') \circ \mathrm{T} \in \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})$, i.e. $\mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})$ is a G'-module. In particular, G' acts on V_λ and by using the fact that $\mathrm{G} \cdot \mathrm{G}'$ acts on W irreducibly, it follows that $\mathrm{W} = \mathrm{V}_\lambda$.
We denote by S the map given by

$$\mathrm{S} : \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W}) \otimes_\mathbb{K} \mathrm{V} \ni \mathrm{T} \otimes v \to \mathrm{T}(v) \in \mathrm{W}.$$

Let $\mathbb{D} := \mathrm{End}_\mathrm{G}(\mathrm{V})$. It is well-known that $\mathbb{D}$ is a division algebra over $\mathbb{K}$. The space V can be seen as $\mathbb{D}$-module where $\mathbb{D}$ acting naturally on the left. Similarly, $\mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})$ can be seen as a right $\mathbb{D}$-module:

$$\mathrm{T} \cdot \mathrm{D} = \mathrm{T} \circ \mathrm{D}\,, \qquad (\mathrm{D} \in \mathbb{D}, \mathrm{T} \in \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})).$$

For every $\mathrm{D} \in \mathbb{D}$, $\mathrm{T} \in \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})$ and $v \in \mathrm{V}$, we get:

$$\mathrm{S}(\mathrm{T} \otimes \mathrm{D} \cdot v) = \mathrm{T}(\mathrm{D}(v)) = \mathrm{T} \circ \mathrm{D}(v) = \mathrm{S}(\mathrm{T} \cdot \mathrm{D} \otimes v)\,.$$

In particular, the map S factors through $\mathrm{V} \otimes_\mathbb{D} \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W})$ and one can easily check that the corresponding map

$$\mathrm{S} : \mathrm{Hom}_\mathrm{G}(\mathrm{V}, \mathrm{W}) \otimes_\mathbb{D} \mathrm{V} \to \mathrm{W}$$

is an isomorphism of $\mathrm{G} \times \mathrm{G}'$-module. The lemma follows.

□

By keeping the notations of Lemma 0.15, we get the following proposition.

Proposition 0.16. *There exists an involution ι on $\mathbb{D}$, a ι-hermitian form γ on* V*, a ι-skew-hermitian form γ' on* V' *such that*

$$\langle u_1 \otimes u'_1, u_2 \otimes u'_2 \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}\left(\gamma(u_1, u_2)\gamma'(u'_1, u'_2)\right)\,, \qquad \left(u_2, u_2 \in \mathrm{V}, u'_1, u'_2 \in \mathrm{V}'\right)\,,$$

and such that $(\mathrm{G}, \mathrm{G}') \approx (\mathrm{G}(\mathrm{V}, \gamma), \mathrm{G}(\mathrm{V}', \gamma'))$.

Before proving this proposition, we say few words about some natural forms that we can construct on the space V. We denote by V the $\mathbb{D}$-vector space and by $\mathrm{V}_\mathbb{K}$ the corresponding vector space over $\mathbb{K}$ (coming from the proof of Lemma 0.15). As explained in the proof of Lemma 0.15, $\mathrm{V}_\mathbb{K}$ is a real subspace of W. On $\mathrm{V}_\mathbb{K}$, we consider the bilinear forms $\beta_{g'_1, g'_2}$, $g'_1, g'_2 \in \mathrm{G}'$, given by:

$$\beta_{g'_1, g'_2}(v_1, v_2) = \langle g'_1 v_1, g'_2 v_2 \rangle\,, \qquad (v_1, v_2 \in \mathrm{V}_\mathbb{K})\ .$$

Lemma 0.17. *For every $g'_1, g'_2 \in \mathrm{G}'$, the forms $\beta_{g'_1, g'_2}$ are* G*-invariant, either zero or non-degenerate. Moreover, there exists $g'_1, g'_2 \in \mathrm{G}'$ such that the form $\beta_{g'_1, g'_2}$ is non-degenerate.*

Let β be a G-invariant bilinear form on $\mathrm{V}_\mathbb{K}$ obtained from Lemma 0.17. We know that $\beta = \beta^\mathrm{s} + \beta^{\mathrm{s-s}}$, with β^s (resp. $\beta^{\mathrm{s-s}}$) is the symmetric (resp. skew-symmetric) bilinear form on $\mathrm{V}_\mathbb{K}$ given by

$$\beta^\mathrm{s}(u, v) = \frac{\beta(u, v) + \beta(v, u)}{2}\,, \qquad \beta^{\mathrm{s-s}}(u, v) = \frac{\beta(u, v) - \beta(v, u)}{2}\,, \qquad (u, v \in \mathrm{V}_\mathbb{K})\ .$$

By using that both β^s and $\beta^{\mathrm{s-s}}$ are G-invariant, it follows that they are either zero or non-degenerate. Using that β is non-degenerate, it follows that at least one of them is non-degenerate.
Let α be such a non-degenerate, either symmetric or skew-symmetric, G-invariant, bilinear form on $\mathrm{V}_\mathbb{K}$, and let $\eta \in \{\pm 1\}$ the element satisfying $\alpha(u, v) = \eta\alpha(v, u)$ for every $u, v \in \mathrm{V}_\mathbb{K}$.

We define $\iota : \mathrm{End}(\mathrm{V}_{\mathbb{K}}) \to \mathrm{End}(\mathrm{V}_{\mathbb{K}})$ the map given by

$$\alpha(\mathrm{T}(u), v) = \alpha(u, \iota(\mathrm{T})v)\,, \qquad (u, v \in \mathrm{V}_{\mathbb{K}})\ .$$

Lemma 0.18. *The map ι is a well-defined involution on $\mathrm{End}(\mathrm{U}_{\mathbb{C}})$ and preserves $\mathbb{D} = \mathrm{End}_{\mathfrak{g}}(\mathrm{V}_{\mathbb{K}})$. Moreover, for every $\mathrm{X}, \mathrm{Y} \in \mathrm{End}(\mathrm{V}_{\mathbb{K}})$, we get:*

$$\iota(\mathrm{XY}) = \iota(\mathrm{Y})\iota(\mathrm{X})\,.$$

Proof. First of all, for every $\mathrm{D} \in \mathrm{End}(\mathrm{V}_{\mathbb{K}})$ and $u, v \in \mathrm{V}_{\mathbb{K}}$, we have:

$$\alpha(u, \mathrm{D}\cdot v) = \eta\alpha(\mathrm{D}\cdot v, u) = \eta\alpha(v, \iota(\mathrm{D})\cdot u) = \alpha(\iota(\mathrm{D})\cdot u, v) = \alpha(u, \iota^2(\mathrm{D})v)\,,$$

i.e. $\iota^2(\mathrm{D}) = \mathrm{D}$.
Moreover, for $\mathrm{D} \in \mathrm{End}_{\mathfrak{g}}(\mathrm{V}_{\mathbb{K}})$ and $\mathrm{X} \in \mathfrak{g}$ (in particular, $\mathrm{DX} = \mathrm{XD}$), we get for every $u, v \in \mathrm{V}_{\mathbb{K}}$ that

$$\begin{aligned} \alpha(u, \iota(\mathrm{D})(\mathrm{X}(v))) &= \alpha(\mathrm{D}\cdot u, \mathrm{X}(v)) = \alpha(\mathrm{X}(\mathrm{D}\cdot u), v) = \alpha(\mathrm{D}(\mathrm{X}\cdot u), v) = \alpha(\mathrm{X}(u), \iota(\mathrm{D})v) \\ &= \alpha(u, \mathrm{X}\iota(\mathrm{D})(v)) = \alpha(u, \mathrm{X}\iota(\mathrm{D})(v))\,, \end{aligned}$$

i.e. $\iota(\mathrm{D})\mathrm{X} = \mathrm{X}\iota(\mathrm{D})$, and in particular, ι preserves $\mathrm{End}_{\mathfrak{g}}(\mathrm{V}_{\mathbb{K}})$ and we get an involution on $\mathbb{D}$.
Finally, for every homogeneous elements $\mathrm{X}, \mathrm{Y} \in \mathrm{End}(\mathrm{V}_{\mathbb{K}})$ and $u, v \in \mathrm{V}_{\mathbb{K}}$, we have:

$$\alpha(u, \iota(\mathrm{XY})v) = \alpha(\mathrm{XY}(u), v) = \alpha(\mathrm{Y}(u), \iota(\mathrm{X})v) = \alpha(u, \iota(\mathrm{Y})\iota(\mathrm{X})v) = \alpha(u, \iota(\mathrm{Y})\iota(\mathrm{X})v)\,,$$

i.e. $\iota(\mathrm{XY}) = \iota(\mathrm{Y})\iota(\mathrm{X})$.

□

We fix a non-degenerate ι-sesquilinear form γ_0 on V, i.e.

$$\gamma_0(ua, vb) = \iota(a)\gamma_0(u, v)b\,, \qquad (u, v \in \mathrm{V}, a, b \in \mathbb{D})\,.$$

We denote by γ_1 the form on $\mathrm{V}_{\mathbb{K}}$ given by

$$\gamma_1(u, v) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\gamma_0(u, v))\ , \qquad (u, v \in \mathbb{V}_{\mathbb{K}})\ .$$

This form is non-degenerate, and in particular, there exists $\mathrm{T} : \mathrm{V}_{\mathbb{K}} \to \mathrm{V}_{\mathbb{K}}$ such that

$$\alpha(u, v) = \gamma_1(\mathrm{T}(u), v)\,, \qquad (u, v \in \mathrm{V}_{\mathbb{K}})\,.$$

Let $\tilde{\gamma}$ be the form on V given by

$$\tilde{\gamma}(u, v) = \gamma_0(\mathrm{T}(u), v)\,, \qquad (u, v \in \mathrm{V})\,.$$

Lemma 0.19. *The form $\tilde{\gamma}$ is non-degenerate, either ι-hermitian or skew-ι-hermitian, and* G*-invariant.*

Proof. First of all, one can notice that T commutes with $\mathbb{D}$. Indeed, for every $u, v \in \mathrm{V}$ and $\mathrm{D} \in \mathbb{D}$, we get:

$$\gamma_1(\mathrm{T}(\mathrm{D}\cdot u), v) = \alpha(\mathrm{D}\cdot u, v) = \alpha(u, \iota(\mathrm{D})\cdot v) = \gamma_1(\mathrm{T}(u), \iota(\mathrm{D})\cdot v) = \gamma_1(\mathrm{D}\cdot \mathrm{T}(u), v)\,,$$

i.e. $\mathrm{T}(\mathrm{D}\cdot u) = \mathrm{D}\cdot\mathrm{T}(u)$ for every $u \in \mathrm{V}$ because γ_1 is non-degenerate.
We know that there exists $\varepsilon \in \{\pm 1\}$ such that $\alpha(u, v) = \varepsilon\alpha(v, u)$ for every $u, v \in \mathrm{V}_{\mathbb{K}}$. We claim that for every $u, v \in \mathrm{V}$,

$$\iota(\tilde{\gamma}(u, v)) = \varepsilon\tilde{\gamma}(v, u)\,.$$

In order to prove the previous equality, it is enough to prove that for every $\mathrm{D} \in \mathbb{D}$, $\mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\mathrm{D}\iota(\tilde{\gamma}(u, v))) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\varepsilon\mathrm{D}\iota(\tilde{\gamma}(v, u)))$ (here, we really need to prove it for every $\mathrm{D} \in \mathbb{D}$, indeed, $\mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\mathrm{X}) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\mathrm{Y})$ does not imply that $\mathrm{X} = \mathrm{Y}$).
For every $u, v \in \mathrm{V}$ and $\mathrm{D} \in \mathbb{D}$, we get:

$$\mathrm{tr}_{\mathbb{D}/\mathbb{C}}(\mathrm{D}\iota(\tilde{\gamma}(u, v))) = \mathrm{tr}_{\mathbb{D}/\mathbb{C}}(\iota(\mathrm{D}\tilde{\gamma}(u, v))) = \mathrm{tr}_{\mathbb{D}/\mathbb{C}}(\tilde{\gamma}(u, v)\iota(\mathrm{D})) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(u, \mathrm{D}\cdot v))$$

$$= \quad \alpha(u, \mathrm{D}\cdot v) = \varepsilon\alpha(\mathrm{D}\cdot v, u) = \varepsilon \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(\mathrm{D}\cdot v, u) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\varepsilon \mathrm{D}\tilde{\gamma}(v, u)),$$

i.e. $\tilde{\gamma}$ is either hermitian or skew-hermitian.
We finish by proving that $\tilde{\gamma}$ is G-invariant. To prove that for every $g \in \mathrm{G}, u, v \in \mathrm{V}$, $\tilde{\gamma}(g(u), g(v)) = \tilde{\gamma}(u, v)$, we are gonna prove that for every $\mathrm{D} \in \mathbb{D}$, $\mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\mathrm{D}\tilde{\gamma}(gu, gv)) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\mathrm{D}\tilde{\gamma}(u, v))$.
We have:

$$\mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\mathrm{D}\tilde{\gamma}(g(u), g(v))) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(\mathrm{D}\cdot g(u), g(v)))) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(g(\mathrm{D}\cdot u), g(v))) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(\mathrm{D}\cdot u, v)) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\mathrm{D}\tilde{\gamma}(u, v)),$$

and the lemma follows.

□

Proof of Proposition 0.16. As explained before, we constructed a G-invariant, non-degenerate, $\pm$-ι-hermitian form γ_2 on V. By using the same method, we construct an involution ι' on D, and a G′-invariant, non-degenerate, $\pm$-ι'-hermitian form γ_2 on V′. Note that $\iota_{|\mathrm{Z}(\mathbb{D})} = \iota'_{|\mathrm{Z}(\mathbb{D})}$, and it follows that there exists $\mathrm{Q} \in \mathbb{D}^*$ such that $\iota' \circ \iota(\mathrm{D}) = \mathrm{QDQ}^{-1}$. We denote by γ' the form on V′ given by

$$\gamma'(u', v') = \gamma_2(u', \mathrm{Q}\cdot v'), \qquad (u', v' \in \mathrm{V}').$$

The form γ' is sesquilinear with respect to the involution ι on $\mathbb{D}$. Indeed, for every $u', v' \in \mathrm{V}'$ and $\mathrm{D} \in \mathbb{D}$, we get:

$$\gamma'(u', \mathrm{D}\cdot v') = \gamma_2(u', \mathrm{Q}\cdot \mathrm{D}v') = \gamma_2(u', \mathrm{QDQ}^{-1}\mathrm{Q}\cdot v') = \gamma_2(u', \iota'(\iota(\mathrm{D})\mathrm{Q}\cdot v') = \gamma_2(u', \mathrm{Q}\cdot v')\iota'^2(\iota(\mathrm{D})) = \gamma'(u', v')\iota(\mathrm{D}).$$

On the tensor product $\mathrm{V} \otimes_{\mathbb{K}} \mathrm{V}'$, we define the form γ_3 by

$$\gamma_3(u_1 \otimes u'_1, u_2 \otimes u'_2) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(u_1, u_2)\gamma'(u'_2, u'_1)), \qquad (u_1, u_2 \in \mathrm{V}, u'_1, u'_2 \in \mathrm{V}').$$

For every $\mathrm{D} \in \mathbb{D}$, $u_1, u_2 \in \mathrm{V}$ and $u'_1, u'_2 \in \mathrm{V}'$, we get:

$$\begin{aligned}\gamma_3(u_1\cdot \mathrm{D} \otimes u'_1, u_2 \otimes u'_2) &= \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(u_1\cdot \mathrm{D}, u_2)\gamma'(u'_2, u'_1)) = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\iota(\mathrm{D})\tilde{\gamma}(u_1, u_2)\gamma'(u'_2, u'_1)) \\ &= \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\tilde{\gamma}(u_1, u_2)\gamma'(u'_2, \mathrm{D}\cdot u'_1)) = \gamma_3(u_1 \otimes \mathrm{D}\cdot u'_1, u_2 \otimes u'_2),\end{aligned}$$

i.e. γ_3 defines a bilinear form $\mathrm{V} \otimes_{\mathbb{D}} \mathrm{V}' = \mathrm{W}$. Note that the form γ_3 is $\mathrm{G}\cdot\mathrm{G}'$-invariant, non-degenerate, and either symmetric or skew-symmetric. In particular, there exists $\mathrm{T} : \mathrm{W} \to \mathrm{W}$ such that for every $w, w' \in \mathrm{W}$,

$$\gamma_3(w, w') = \langle \mathrm{T}(w), w' \rangle, \qquad (w, w' \in \mathrm{W}).$$

By using the $\mathrm{G}\cdot\mathrm{G}'$-invariance of γ_3, it follows that T commutes with both G and G′. Because the action of $\mathrm{G}\cdot\mathrm{G}'$ on W is irreducible, it follows that $\mathrm{T} = \mathrm{D}$, with $\mathrm{D} \in \mathbb{D}^*$. Let $\varepsilon \in \{\pm 1\}$ such that $\gamma_3(w, w') = \varepsilon\gamma_3(w', w)$ for every $w, w' \in \mathrm{W}$. Then, $\varepsilon\langle \mathrm{D}\cdot w', w\rangle = \langle w, \iota(\mathrm{D})\cdot w'\rangle$, and by replacing the form $\tilde{\gamma}$ by the form γ on V given by

$$\gamma(v_1, v_2) = \tilde{\gamma}(v_1 \cdot \mathrm{D}^{-1}, v_2), \qquad (v_1, v_2 \in \mathrm{V}),$$

we get that

$$\langle (u_1 \otimes u'_1, u_2 \otimes u'_2 \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{K}}(\gamma(u_1, u_2)\gamma'(u'_2, u'_1)), \qquad (u_1, u_2 \in \mathrm{V}, u'_1, u'_2 \in \mathrm{V}'),$$

and $\mathrm{G} \subseteq \mathrm{G}_{\mathbb{D}}(\mathrm{V}, \gamma), \mathrm{G}' \subseteq \mathrm{G}_{\mathbb{D}}(\mathrm{V}', \gamma')$. By using that $\mathrm{G}_{\mathbb{D}}(\mathrm{V}, \gamma), \mathrm{G}_{\mathbb{D}}(\mathrm{V}', \gamma') \subseteq \mathrm{Sp}(\mathrm{W})$ and that G (resp. G′) commutes with $\mathrm{G}_{\mathbb{D}}(\mathrm{V}', \gamma')$ (resp. $\mathrm{G}_{\mathbb{D}}(\mathrm{V}, \gamma)$), it follows that $(\mathrm{G}, \mathrm{G}') = (\mathrm{G}_{\mathbb{D}}(\mathrm{V}, \gamma), \mathrm{G}_{\mathbb{D}}(\mathrm{V}', \gamma'))$ and the proposition is proven.

□

Corollary 0.20. *Let* $(\mathrm{G}, \mathrm{G}')$ *be an irreducible reductive dual pair of type I. Then,*

(1) If $\mathbb{K} = \mathbb{C}$, $(\mathrm{G}, \mathrm{G}') = (\mathrm{Sp}(2n, \mathbb{C}), \mathrm{O}(m, \mathbb{C})) \subseteq \mathrm{Sp}(2nm, \mathbb{C})$,
(2) If $\mathbb{K} = \mathbb{R}$, $(\mathrm{G}, \mathrm{G}')$ *is isomorphic to one of the following pair:*

(a) $(\mathrm{Sp}(2n,\mathbb{R}),\mathrm{O}(p,q,\mathbb{R}))\subseteq\mathrm{Sp}(2n(p+q),\mathbb{R})$,
(b) $(\mathrm{Sp}(2n,\mathbb{C}),\mathrm{O}(m,\mathbb{C}))\subseteq\mathrm{Sp}(4nm,\mathbb{R})$,
(c) $(\mathrm{U}(p,q,\mathbb{C}),\mathrm{U}(r,s,\mathbb{C}))\subseteq\mathrm{Sp}(2(p+q)(r+s),\mathbb{R})$,
(d) $(\mathrm{Sp}(p,q,\mathbb{H}),\mathrm{O}^*(m,\mathbb{H}))\subseteq\mathrm{Sp}(4m(p+q),\mathbb{R})$.

We now prove a similar result for the dual pairs of type II.

Proposition 0.21. *Let* (G,G') *be an irreducible reductive dual pair of type II in* $\mathrm{Sp}(\mathrm{W})$. *Then,* $\mathrm{W}=\mathrm{U}\oplus\mathrm{U}^*$, *with* U *(resp.* U^**) is* $\mathrm{G}\cdot\mathrm{G}'$*-invariant and irreducible, and there exists a division algebra* $\mathbb{D}$ *over* $\mathbb{K}$*, a right vector space* V *over* $\mathbb{D}$*, a left vector space* V' *over* $\mathbb{D}$ *such that*

$$\mathrm{W}=\mathrm{V}\otimes_{\mathbb{D}}\mathrm{V}'$$

such that $\mathrm{W}=\mathrm{V}\otimes_{\mathbb{D}}\mathrm{V}'\oplus(\mathrm{V}\otimes_{\mathbb{D}}\mathrm{V}')^*$ *and* $(\mathrm{G},\mathrm{G}')=(\mathrm{GL}_{\mathbb{D}}(\mathrm{V}),\mathrm{GL}_{\mathbb{D}}(\mathrm{V}'))$.

Proof. The proof of the existence of $\mathbb{D}$, V and V' is similar to the one of Lemma 0.15. Obviously, $\mathrm{GL}_{\mathbb{D}}(\mathrm{V})\subseteq\mathrm{Sp}(\mathrm{W})$ (resp. $\mathrm{GL}_{\mathbb{D}}(\mathrm{V}')\subseteq\mathrm{Sp}(\mathrm{W})$) and commute with G' (resp G) and it follows that $(\mathrm{G},\mathrm{G}')=(\mathrm{GL}_{\mathbb{D}}(\mathrm{V}),\mathrm{GL}_{\mathbb{D}}(\mathrm{V}'))$.

□

Corollary 0.22. *Let* (G,G') *be an irreducible reductive dual pair of type II. Then,*

(1) If $\mathbb{K}=\mathbb{C}$, $(\mathrm{G},\mathrm{G}')=(\mathrm{GL}(n,\mathbb{C}),\mathrm{GL}(m,\mathbb{C}))\subseteq\mathrm{Sp}(2nm,\mathbb{C})$,
(2) If $\mathbb{K}=\mathbb{R}$, (G,G') *is isomorphic to one of the following pair:*
  (a) $(\mathrm{GL}(n,\mathbb{R}),\mathrm{GL}(m,\mathbb{R}))\subseteq\mathrm{Sp}(2nm,\mathbb{R})$,
  (b) $(\mathrm{GL}(n,\mathbb{C}),\mathrm{GL}(m,\mathbb{C}))\subseteq\mathrm{Sp}(4nm,\mathbb{R})$,
  (c) $(\mathrm{GL}(n,\mathbb{H}),\mathrm{GL}(m,\mathbb{H}))\subseteq\mathrm{Sp}(8nm,\mathbb{C})$.

References

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