Polynomial Invariants and Character Varieties of Classical Groups

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Contents

1 Character Varieties

Let G be a complex reductive algebraic group and Γ be a finitely generated group with r-generators, i.e. $\Gamma = \langle \gamma_1, \dots, \gamma_r, \mathrm{R}_i(\gamma_1, \dots, \gamma_r) = 1, i \in \mathrm{I}\rangle$.
We denote by $\mathrm{Hom}(\Gamma, \mathrm{G})$ the set of morphisms $\rho : \Gamma \to \mathrm{G}$. We have a natural action φ of G on $\mathrm{Hom}(\Gamma, \mathrm{G})$ given by:

$$\varphi(g)(\rho)(\gamma) = g\rho(\gamma)g^{-1}, \qquad (g \in \mathrm{G}, \rho \in \mathrm{Hom}(\Gamma, \mathrm{G}), \gamma \in \Gamma).$$

The character variety of (Γ, G), denoted by $\mathscr{X}(\Gamma, \mathrm{G})$, is defined as:

$$\mathscr{X}(\Gamma, \mathrm{G}) = \mathrm{Hom}(\Gamma, \mathrm{G})//\mathrm{G}$$

where we denote by // the GIT-quotient or categorical quotient, i.e.

$$\mathscr{X}(\Gamma, \mathrm{G}) = \mathrm{Spec}_{\mathrm{max}}(\mathbb{C}[\mathrm{Hom}(\Gamma, \mathrm{G})]^{\mathrm{G}}).$$

Remark 1.1. In the definition of the character variety, we used the GIT-quotient instead of the classical quotient. Let's see the difference between those two quotients on examples.

Example 1.2. 1. Let $G = \mathbb{C}^*$, $X = \mathbb{C}^2$ and $G \curvearrowright X$ given by

$$\lambda \cdot (x, y) = (\lambda x, \lambda^{-1} y), \qquad (\lambda \in \mathbb{C}^*, (x, y) \in \mathbb{C}^2).$$

We want to determine explicitly X/G. How can we parametrise the orbits?

Clearly, let $(x, y) \in \mathbb{C}^2$. Then $\mathcal{O}(x, y) = \{(\lambda x, \lambda^{-1} y), \lambda \in \mathbb{C}^*\}$. In particular, $xy = \lambda x \lambda^{-1} y$ and then if $xy \neq 0$, we have $(x_1, y_1) \in \mathcal{O}(x, y)$ if and only if $x_1 y_1 = xy$. What happen if $xy = 0$? We have three different orbits:

$$\mathcal{O}_0 = \{(0,0)\} \qquad \mathcal{O}_1 = \{(x, 0), x \in \mathbb{C}^*\} \qquad \mathcal{O}_1 = \{(0, y), y \in \mathbb{C}^*\}$$

The orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ are not closed. Indeed, the sequence $x_n = (\frac{1}{n}, 0) \in \mathcal{O}_1$ and $x_n \to (0, 0) \notin \mathcal{O}_1$ when n goes to infinity. In particular, the quotient X/G is not Hausdorff.

Let's now determine the GIT-quotient. For that, we have to determine the ring $\mathbb{C}[X]^G$, i.e. the set of polynomials $P : X \to \mathbb{C}$ such that

$$P(\lambda \cdot (x, y)) = P((x, y))$$

i.e. $P((\lambda x, \lambda^{-1} y)) = P((x, y)$. One can check easily that $\mathbb{C}[X]^G = \mathbb{C}[xy]$, and then:

$$\mathrm{Spec}(\mathbb{C}[X]^G) = \mathrm{Spec}(\mathbb{C}[xy]) = \mathbb{C}.$$

2. Let $\Gamma = \mathbb{Z}$ and $G = SL(2, \mathbb{C})$. Obviously, $\mathrm{Hom}(\mathbb{Z}, G) \approx G$ and $\mathrm{Hom}(\mathbb{Z}, G)/G$ is the set of conjugacy classes with respect to the action of G on itself by conjugation. We denote by A the following matrix of $SL(2, \mathbb{C})$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Then for all $t \neq 0$, we have:

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}^{-1} = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}$$

and we have $\lim_{t \to 0} \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix} = \mathrm{Id}_2$. Again, some orbits are open in $SL(2, \mathbb{C}/SL(2, \mathbb{C})$. In Section 2, we will determine explicitly $\mathrm{Hom}(F_i, SL(2, \mathbb{C}))//SL(2, \mathbb{C})$ for $i = 1, 2$.

Remark 1.3. We denote by $X_1, \ldots, X_r$ the generators of the free group F_r. Then, the map:

$$\mathrm{Hom}(F_r, G) \ni \rho \to (\rho(X_1), \ldots, \rho(X_r)) \in G^r$$

is an isomorphism.

2 $SL(2, \mathbb{C})$-character varieties

Let's start with $\Gamma = \mathbb{Z}$ and $G = SL(2, \mathbb{C})$.

Theorem 1

Let $f \in \mathbb{C}[SL(2, \mathbb{C})]^{SL(2, \mathbb{C})}$. There exists a polynomial $F \in \mathbb{C}[t]$ such that $f(g) = F(\mathrm{tr}(g))$.

Proof. In $\mathrm{SL}(2,\mathbb{C})$, the eigenvalues of a matrix g only depends of the trace. Indeed, if we denote by λ_1, λ_2 the two eigenvalues, we have:

$$\mathrm{tr}(g) = \lambda_1 + \lambda_2 \qquad 1 = \det(g) = \lambda_1\lambda_2.$$

In particular, we have:

$$\lambda_1 = \frac{\mathrm{tr}(g) + \sqrt{\mathrm{tr}(g)^2 - 4}}{2}, \qquad \lambda_2 = \frac{\mathrm{tr}(g) - \sqrt{\mathrm{tr}(g)^2 - 4}}{2},$$

and if $|\mathrm{tr}(g)| \neq 2$, the matrix g is diagonalizable and is in the orbit of the matrix $\mathrm{M}_{\mathrm{tr}(g)}$, where M_t is the matrix defined by

$$\mathrm{M}_t = \begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix}$$

If $g \in \mathrm{SL}(2,\mathbb{C})$ such that $|\mathrm{tr}(g)| \neq 2$, we define the function F as:

$$\mathrm{F}(\mathrm{tr}(g)) = f(g).$$

If $g \in \mathrm{SL}(2,\mathbb{C})$ such that $\mathrm{tr}(g) = \pm 2$, we have two possibilities: either g is diagonalisable and $g = \pm\mathrm{Id}$ or g is non-diagonalisable, and according to the Jordan decomposition, g in conjugate to the matrix $\mathrm{M}_{\mathrm{tr}(g)}$. We extend the function F on the set $\{g \in \mathrm{SL}(2,\mathbb{C}), |\mathrm{tr}(g)| = 2 \text{ and g not diagonalisable}\}$ by $f(g) = \mathrm{F}(\mathrm{tr}(g))$ and we finally defined F everywhere by noticing that

$$\mathrm{Id} \in \overline{\mathscr{O}(\mathrm{M}_2)}, \qquad \text{and } -\mathrm{Id} \in \overline{\mathscr{O}(\mathrm{M}_{-2})}.$$

□

Corollary 1

The character variety $\mathscr{X}(\mathbb{Z}, \mathrm{SL}(2,\mathbb{C}))$ is isomorphic to $\mathbb{C}$.

Proof. Let I be a maximal ideal of $\mathbb{C}[\mathrm{Hom}(\Gamma,\mathrm{G})]^{\mathrm{G}} \cong \mathbb{C}[z]$. Using that $\mathbb{C}[z]$ is euclidian, there exists $\mathrm{P} \in \mathbb{C}[z]$ such that $\mathrm{I} = (\mathrm{P})$. Because I is maximal, it follows that P is irreducible. In particular, there exists $z_0 \in \mathbb{C}$ such that $\mathrm{P}(z) = z - z_0$. The map $\mathbb{C}[z] \ni (z - z_0) \to z_0 \in \mathbb{C}$ gives the expected isomorphism.

□

We now deal with the case $\Gamma = \mathrm{F}_2$.

Theorem 2 – Fricke-Vogt

Let $f : \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to \mathbb{C}$ *be a regular function which is invariant under the action of* $\mathrm{SL}(2,\mathbb{C})$*, i.e.*

$$f(u,v) = f(g\cdot(u,v)) = f(gug^{-1}, gvg^{-1}).$$

Then there exists a polynomial function $\mathrm{F} \in \mathbb{C}[x,y,z]$ *such that:*

$$f(\xi,\eta) = \mathrm{F}(\mathrm{tr}(\xi), \mathrm{tr}(\eta), \mathrm{tr}(\xi\eta)).$$

Proof. See [1].

□

Corollary 2

The character variety $\mathscr{X}(\mathrm{F}_2, \mathrm{SL}(2,\mathbb{C}))$ is $\mathbb{C}^3$.

Proof. According to theorem 2, the space of $\mathrm{G} = \mathrm{SL}(2,\mathbb{C})$-invariants $\mathbb{C}[\mathrm{G}\times\mathrm{G}]^{\mathrm{G}}$ is a quotient of $\mathbb{C}[x, y, z]$. To prove the equality, we need to prove that the map

$$\tau : \mathrm{G}\times\mathrm{G} \ni (\xi, \eta) \to (\mathrm{tr}(\xi), \mathrm{tr}(\eta), \mathrm{tr}(\xi\eta)) \in \mathbb{C}^3$$

is surjective. In particular, we want to prove that for every $(x, y, z) \in \mathbb{C}^3$, there exists $(\xi, \eta) \in \mathrm{G}\times\mathrm{G}$ such that $(x, y, z) = (\mathrm{tr}(\xi), \mathrm{tr}(\eta), \mathrm{tr}(\xi\eta))$.
For that, we choose $\rho \in \mathbb{C}$ such that $\rho + \rho^{-1} = z$ and let ξ_x and $\eta_{y,\rho}$ the matrices of $\mathrm{SL}(2,\mathbb{C})$ given by

$$\xi_x = \begin{pmatrix} x & -1 \\ 1 & 0 \end{pmatrix}, \qquad \eta_{y,\rho} = \begin{pmatrix} 0 & \rho^{-1} \\ \rho & y \end{pmatrix}.$$

One can easily see that $\tau(\xi_x, \eta_{y,\rho}) = (x, y, z)$.

□

3 The algebra of full traces

We denote by $\widehat{\mathrm{G}}$ the set of finite dimensional irreducible representations of G. For every $\gamma \in \Gamma$ and $(\phi, \mathrm{V}_\phi) \in \widehat{\mathrm{G}}$, we denote by $\tau_{\gamma,\phi}$ the map

$$\tau_{\gamma,\phi} : \mathrm{Hom}(\Gamma, \mathrm{G}) \ni \rho \to \mathrm{tr}(\phi(\rho(\gamma))) \in \mathbb{C}.$$

We denote by $\mathscr{F}\mathscr{T}(\Gamma, \mathrm{G})$ the complex algebra generated by the elements $\tau_{\gamma,\phi}, \gamma \in \Gamma, (\phi, \mathrm{V}_\phi) \in \widehat{\mathrm{G}}$. One can easily see that

$$\mathscr{F}\mathscr{T}(\Gamma, \mathrm{G}) \subseteq \mathbb{C}[\mathscr{X}(\Gamma, \mathrm{G})].$$

Question 1

Is the previous inclusion an equality?

In this note, we are going to see that the answer is the previous inclusion is an equality for $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SL}(n,\mathbb{C})$, $\mathrm{Sp}(2n,\mathbb{C})$, $\mathrm{SO}(2n+1,\mathbb{C})$ but the inclusion is proper for $\mathrm{G} = \mathrm{SO}(2n,\mathbb{C})$.

Let G be a **simply connected reductive group** of rank n. We denote by $\omega_1, \ldots, \omega_n$ the fundamental weights of G and by (π_k, V_{ϕ_k}) the corresponding irreducible representations of weight ω_k.

Proposition 1

For every irreducible finite dimensional representation (Π, V) *of* G*, there exists a polynomial* $P \in \mathbb{C}[X_1, \ldots, X_n])$ *with coefficients in* $\mathbb{Z}_+$ *such that*

$$\mathrm{tr}(\Pi(g)) = P(\mathrm{tr}(\pi_1(g)), \ldots, \mathrm{tr}(\pi_n(g))), \qquad (g \in G).$$

Proof. Let Π be such a representation. The representation Π is an highest weight module and the corresponding weight ω is of the form:

$$\omega = \sum_{k=1}^{n} \alpha_i \omega_i, \qquad \alpha_i \in \mathbb{Z}_+.$$

We will proceed by induction. Il $\alpha_i = 1$ and $\alpha_j = 0$ if $i \neq j$, then the result is trivial: the polynomial P is $P(X_1, \ldots, X_n) = X_i$.

For a general ω, we know that the representation (E, ρ)

$$E = V_{\phi_1}^{\otimes \alpha_1} \otimes \ldots \otimes V_{\phi_n}^{\otimes \alpha_n}$$

contains a unique copie of (V, Π). Moreover, we have:

$$E = V \oplus W,$$

where (W, η) contains only highest weight representations whose weights $\gamma = \sum_{k=1}^{n} \beta_k \omega_k$ verify $\beta_k \leq \alpha_k$. Then, there exists a polynomial Q with coefficient in $\mathbb{Z}_+$ such that:

$$\mathrm{tr}(\eta(g)) = Q(\mathrm{tr}(\pi_1(g)), \ldots, \mathrm{tr}(\pi_n(g))).$$

The character of the representation $\rho(g)$ is given by

$$\mathrm{tr}(\rho(g)) = \prod_{k=1}^{n} \mathrm{tr}(\pi_k(g))^{\alpha_k},$$

and then we got

$$\mathrm{tr}(\Pi(g)) = \prod_{k=1}^{n} \mathrm{tr}(\pi_k(g))^{\alpha_k} - Q(\mathrm{tr}(\pi_1(g)), \ldots, \mathrm{tr}(\pi_n(g))).$$

The proposition is proved.

□

For a finitely generated group Γ, we denote by $\mathscr{T}(\Gamma, G) \subseteq \mathscr{F}\mathscr{T}(\Gamma, G)$ the subalgebra generated by the function $\tau_{\gamma,\pi_k}, \gamma \in \Gamma, k \in \{1, \ldots, n\}$.

Corollary 3

We have the following equality:

$$\mathscr{F}\mathscr{R}(\Gamma, G) = \mathscr{F}\mathscr{T}(\Gamma, G).$$

4 Polynomial invariants of classical complex groups

4.1 $\mathrm{G} = \mathrm{GL}(n, \mathbb{C})$

Let V be a complex vector space and GL(V) the corresponding general linear group. Let π the natural action of G on V. We denote by π^* the contragredient representation and by Ad the action of G on $\mathfrak{g} = \mathrm{End}(\mathrm{V})$.
We want to determine the generators of

$$\mathbb{C}[n\mathrm{End}(\mathrm{V})]^{\mathrm{GL}(\mathrm{V})}, \qquad (n \in \mathbb{N}).$$

We have the decomposition

$$\mathbb{C}[n\mathrm{End}(\mathrm{V})] = \bigoplus_{k=0}^{\infty} \mathbb{C}^k[n\mathrm{End}(\mathrm{V})],$$

and one can easily see that

$$\mathbb{C}[n\mathrm{End}(\mathrm{V})]^{\mathrm{GL}(\mathrm{V})} = \bigoplus_{k=0}^{\infty} \mathbb{C}^k[n\mathrm{End}(\mathrm{V})]^{\mathrm{GL}(\mathrm{V})},$$

Let P be an element of $\mathbb{C}^k[n\mathrm{End}(\mathrm{V})]^{\mathrm{GL}(\mathrm{V})}$. The polarization $\mathrm{Q_P}$ of P is given by

$$\mathrm{Q_P}(\mathrm{A}^1, \ldots, \mathrm{A}^k) = \frac{1}{k!}\frac{\partial}{\partial\lambda_1}\cdots\frac{\partial}{\partial\lambda_k}\mathrm{P}(\sum_{i=1}^{k}\lambda_i \mathrm{A}^i)_{|\lambda=0},$$

where $\mathrm{A}^i = (\mathrm{A}^i_1, \ldots, \mathrm{A}^i_n) \in n\mathrm{End}(V)$. The element $\mathrm{Q_P}$ is a multilinear map

$$\mathrm{Q_P} : \underbrace{n\mathrm{End}(\mathrm{V}) \times \ldots \times n\mathrm{End}(\mathrm{V})}_{k-\text{times}} \to \mathbb{C}.$$

In particular, we see that $\mathrm{Q_P}$ is of the form

$$\mathrm{Q_P} = \sum_{i_1,\ldots,i_k \in \{1,\ldots,n\}} c_{i_1,\ldots,i_k} \mathrm{U} \circ \iota_{i_1,\ldots,i_k},$$

where $c_{i_1,\ldots,i_k} \in \mathbb{C}$, $\mathrm{U} \in \mathrm{Mult}(k\mathrm{End}(V), \mathbb{C})$ and

$$\iota_{i_1,\ldots,i_k} : n\mathrm{End}(V)^{\times k} \ni (A^1, \ldots, A^k) \to (A^1_{i_1}, \ldots, A^k_{i_k}) \in \mathrm{End}(V)^{\times k}.$$

Note that some of the constants $c_{i_1,\ldots,i_k}$ are zero. In particular, $\mathrm{U} \in \mathrm{End}(\mathrm{V})^{\otimes k^*} \approx \mathrm{End}(\mathrm{V})^{*\otimes k}$ and then we have to determine

$$\mathrm{End}(\mathrm{V})^{\otimes k^* \mathrm{GL}(\mathrm{V})}.$$

We denote by

$$\Gamma : \mathrm{V}^* \otimes \mathrm{V} \ni v^* \otimes v \to \Gamma(v^* \otimes v) \in \mathrm{End}(\mathrm{V}) \tag{1}$$

the map given by $\Gamma(v^* \otimes v)(w) = v^*(w)v$. The map Γ is an isomorphism of GL(V)-module.
It is known that $\mathrm{End}(\mathrm{V})^{\mathrm{GL}(\mathrm{V})} = \{\alpha \mathrm{Id}_V, \alpha \in \mathbb{C}\}$. Then we have $\mathbb{C} \cdot \Gamma^{-1}(\mathrm{Id_V}) = (\mathrm{V}^* \otimes \mathrm{V})^{\mathrm{GL}(\mathrm{V})}$. We denote by $\theta_1 = \Gamma^{-1}(\mathrm{Id}_V)$. If we fix a basis $\{v_1, \ldots, v_n\}$ of V and $\{v_1^*, \ldots, v_n^*\}$ the dual basis, we get that:

$$\theta_1 = \sum_{k=1}^{n} v_k^* \otimes v_k.$$

We have similar results for the invariants for the action of GL(V) on $(V^* \otimes V)^{\otimes k}$. We denote by $\theta_k = \otimes^k \theta_1$. We have a natural action of $\mathscr{S}_k \times \mathscr{S}_k$ on the space $(V^* \otimes V)^{\otimes k}$ (where the first copy of $\mathscr{S}_k$ acts on $V^{*\otimes k}$ and the second copie on $V^{\otimes k}$).

Theorem 3 – H. Weyl [5]

We get $\left((V^* \otimes V)^{\otimes k}\right)^{GL(V)} = (\mathscr{S}_k \times \mathscr{S}_k) \cdot \theta_k$.

Now let's consider a more general action for GL(V). As explained in equation (1), we have the following isomorphisms of GL(V)-modules:

$$\mathrm{End}(V) \cong V^* \otimes V \cong (V^* \otimes V)^* \,.$$

Then for every $k \in \mathbb{N}$, we got the GL(V)-modules isomorphisms:

$$\mathrm{End}(V)^{\otimes k} \cong \left((V^* \otimes V)^{\otimes k}\right)^* \cong (V^* \otimes V)^{\otimes k} \cong V^{*\otimes k} \otimes V^{\otimes k} \cong (V^{*\otimes k} \otimes V^{\otimes k})^* \,.$$

We consider here the action of GL(V) on the space $(V^{*\otimes k} \otimes V^{\otimes k})^*$. A linear form λ on $V^{*\otimes k} \otimes V^{\otimes k}$ is GL(V)-invariant if $g \cdot \lambda = \lambda$, i.e.

$$\lambda(\pi^*(g)\phi_1 \otimes \ldots \otimes \pi^*(g)\phi_k \otimes \pi(g)v_1 \otimes \ldots \otimes \pi(g)v_k) = \lambda(\phi_1 \otimes \ldots \otimes \phi_k \otimes v_1 \otimes \ldots \otimes v_k) \,,$$

where $v_1, \ldots, v_k \in V$ and $\phi_1, \ldots, \phi_k \in V^*$.
We can write the invariants we got in a different way ([2]). To keep the notations of [2], we will consider the action of GL(V) on the space $(V^{*\otimes k} \otimes V^{\otimes k})^*$. According to the previous paragraph, the forms $\lambda_\sigma, \sigma = \sigma_1 \times \sigma_2 \in \mathscr{S}_k \times \mathscr{S}_k$ given by:

$$\lambda_\sigma(\phi_1 \otimes \ldots \otimes \phi_k \otimes v_1 \otimes \ldots \otimes v_k) = \phi_{\sigma_1^{-1}(1)} \otimes \ldots \otimes \phi_{\sigma_1^{-1}(k)}(v_{\sigma_2^{-1}(1)} \otimes \ldots \otimes v_{\sigma_2^{-1}(k)})$$

are G-invariants in $(V^{*\otimes k} \otimes V^{\otimes k})^*$. We have:

$$\begin{aligned}
\lambda_\sigma(\phi_1 \otimes \ldots \otimes \phi_k \otimes v_1 \otimes \ldots \otimes v_k) &= \phi_{\sigma_1^{-1}(1)} \otimes \ldots \otimes \phi_{\sigma_1^{-1}(k)}(v_{\sigma_2^{-1}(1)} \otimes \ldots \otimes v_{\sigma_2^{-1}(k)}) \\
&= \prod_{i=1}^{k} \phi_{\sigma_1^{-1}(i)}(X_{\sigma_1^{-1}(i)}) = \prod_{i=1}^{k} \phi_{\sigma_2\sigma_1^{-1}(i)}(v_i) \\
&= \mu_{\sigma_2\sigma_1^{-1}}(\phi_1 \otimes \ldots \otimes \phi_k \otimes X_1 \otimes \ldots \otimes v_k)
\end{aligned}$$

Obviously, the form μ_σ (and similarly λ_σ) can be define using an element in $\mathscr{S}_n$ by:

$$\mu_\sigma(\phi_1 \otimes \ldots \otimes \phi_k \otimes v_1 \otimes \ldots \otimes v_k) = \prod_{i=1}^{k} \phi_{\sigma(i)}(v_i) \,.$$

Remark 4.1. Let $\Gamma : V^* \otimes V \to \mathrm{End}(V)$ the GL(V)-equivariant isomorphism considered before. Let $\phi \otimes v$ be an element of $V^* \otimes V$. Then the trace of the endomorphism $\Gamma(\phi \otimes v)$ is $\phi(v)$. Indeed, let $\{v_1, \ldots, v_k\}$ as before. Then

$$\phi = \sum_{i=1}^{k} a_i v_i^* \qquad v = \sum_{i=1}^{k} b_i v_i \,.$$

We have

$$\Gamma(\phi \otimes v)(v_i) = \phi(v_i)v = \sum_{h=1}^{k} b_h \phi(v_i) v_h \,,$$

and then

$$\mathrm{tr}(\Gamma(\phi \otimes v)) = \sum_{h=1}^{k} \mathrm{pr}_{\mathbb{C}v_h} \Gamma(\phi \otimes v)(v_h) = \sum_{h=1}^{k} b_h \phi(v_h) = \sum_{h=1}^{k} a_h b_h = \phi(v)\,.$$

Similarly, we have a natural multiplication on End(V). We have the following result:

$$\Gamma(\phi_1 \otimes v_1)\Gamma(\phi_2 \otimes v_2) = \Gamma(\phi_1 \otimes \phi_2(v_1)v_2)\,.$$

In particular, we have:

$$\mathrm{tr}(\Gamma(\phi_1 \otimes v_1)\Gamma(\phi_2 \otimes v_2)) = \mathrm{tr}(\Gamma(\phi_1 \otimes \phi_2(v_1)v_2)) = \phi_2(v_1)\phi_1(v_2)\,.$$

and more generally

$$\mathrm{tr}\left(\prod_{i=1}^{k} \Gamma(\phi_i \otimes v_i)\right) = \prod_{i=1}^{k} \phi_{i+1}(v_i)\,,$$

where, by convention, we have $v_1 = v_{k+1}$.

The form μ_σ is a GL(V)-invariant element of $\left(\mathrm{V}^{*\otimes k} \otimes V^{\otimes k}\right)^*$. Using that

$$\left(\mathrm{V}^{*\otimes k} \otimes V^{\otimes k}\right)^* \approx \mathrm{End(V)}^{\otimes k^*}\,,$$

we get a form $\mu_\sigma : \mathrm{End(V)} \otimes \ldots \otimes \mathrm{End(V)} \to \mathbb{C}$ given for $\mathrm{A}_i = \Psi(\phi_i \otimes v_i)\,, \phi_i \in \mathrm{V}^*, v_i \in \mathrm{V}\,, i = 1\,, \ldots\,, k$, by

$$\mu_\alpha(\Gamma(\phi_1 \otimes v_1), \ldots, \Gamma(\phi_k \otimes v_k)) = \mu_\sigma(\phi_1 \otimes \phi_2 \otimes \ldots \otimes \phi_k \otimes v_1 \otimes \ldots \otimes v_k)\,.$$

Remark 4.2. Every matrix $\mathrm{A} \in \mathrm{End(V)}$ can be written as a finite sum of elements of the form $\Gamma(\phi \otimes v)$ so by using that μ_σ is multilinear, we can define μ_σ for a general element $\mathrm{A}_1 \otimes \ldots \otimes \mathrm{A}_k$, with $\mathrm{A}_1, \ldots, \mathrm{A}_k \in$ End(V).

Proposition 2

Let σ be an element of $\mathscr{S}_k$ and $\sigma = (i_1 \ldots i_a)(j_1 \ldots j_b) \ldots (t_1 \ldots t_c)$ a decomposition of σ in disjoint cycles including the ones of length one. For every $\mathrm{A}_1, \ldots, \mathrm{A}_k \in \mathrm{End(V)}$, we have:

$$\mu_\sigma(\mathrm{A}_1 \otimes \ldots \otimes \mathrm{A}_k) = \mathrm{tr}(\mathrm{A}_{i_1} \ldots \mathrm{A}_{i_a})\mathrm{tr}(\mathrm{A}_{j_1} \ldots \mathrm{A}_{j_b}) \ldots \mathrm{tr}(\mathrm{A}_{t_1} \ldots \mathrm{A}_{t_c})$$

Proof. We first assume that σ is a r-cycle, with $r \leq k$. Note $\sigma = (i_1 i_2 \ldots i_r)$. Then

$$\mathrm{Supp}(\sigma) = \{i_1\,, \ldots\,, i_r\}\,, \qquad \mathrm{Fix}(\sigma) = \{1\,, \ldots\,, k\} \setminus \mathrm{Supp}(\sigma)\,.$$

We get

$$\begin{aligned}
\mu_\alpha(\Gamma(\phi_1 \otimes v_1) \otimes \ldots \otimes \Gamma(\phi_k \otimes v_k)) &= \prod_{i=1}^{k} \phi_{\sigma_i^{-1}}(v_i) \\
&= \prod_{i \in \mathrm{Supp}(\sigma)} \phi_{\sigma_i^{-1}}(v_i) \prod_{i \in \mathrm{Fix}(\sigma)} \phi_{\sigma_i^{-1}}(v_i) \\
&= \prod_{i \in \mathrm{Supp}(\sigma)} \phi_{\sigma_i^{-1}}(v_i) \prod_{i \in \mathrm{Fix}(\sigma)} \phi_i(v_i) \\
&= \prod_{i \in \mathrm{Supp}(\sigma)} \phi_{\sigma_i^{-1}}(v_i) \prod_{i \in \mathrm{Fix}(\sigma)} \mathrm{tr}(\Gamma(\phi_i \otimes v_i)) .
\end{aligned}$$

We now need to simplify $\prod_{i \in \mathrm{Supp}(\sigma)} \phi_{\sigma_i^{-1}}(v_i)$. We get

$$\begin{aligned}
\prod_{i \in \mathrm{Supp}(\sigma)} \phi_{\sigma_i^{-1}}(v_i) &= \phi_{i_1}(v_{i_1})\phi_{i_2}(v_{i_3}) \ldots \phi_{i_r}(v_{i_1}) \\
&= \mathrm{tr}(\Gamma(\phi_1 \otimes v_1)\Gamma(\phi_2 \otimes v_2) \ldots \Gamma(\phi_r \otimes v_r)) ,
\end{aligned}$$

so the proposition follows for an r-cycle σ. Every permutation can be written as a product $\sigma = \sigma_1 \circ \ldots \circ \sigma_m$ with $\mathrm{Supp}(\sigma_i) \cap \mathrm{Supp}(\sigma_j) = \emptyset$ for $i \neq j$ (note that some σ_i's can be of length one). In particular, we get

$$\mu_\alpha(\Gamma(\phi_1 \otimes v_1) \otimes \ldots \otimes \Gamma(\phi_k \otimes v_k)) = \prod_{c=1}^{m} \left(\prod_{i \in \mathrm{Supp}(\sigma_c)} \phi_{\sigma_i^{-1}}(v_i) \right) ,$$

and then we get, for every $1 \leq c \leq m$, that

$$\prod_{i \in \mathrm{Supp}(\sigma_c)} \phi_{\sigma_i^{-1}}(v_i) = \mathrm{tr}(\Gamma(\phi_{i_1^c} \otimes v_{i_1^c})\Gamma(\phi_{i_2^c} \otimes v_{i_2^c}) \ldots \Gamma(\phi_{i_{r_c}^c} \otimes v_{i_{r_c}^c})) ,$$

with $\sigma_c = (i_1^c i_2^c \ldots i_{r_c}^c)$. The proposition follows. □

Theorem 4 – Procesi

The algebra $\mathbb{C}[n\mathrm{End(V)}]^{\mathrm{GL(V)}}$ *is generated by elements of the form*

$$(\mathrm{A}_1 , \ldots , \mathrm{A}_n) \to \mathrm{tr}(\mathrm{P}(\mathrm{A}_1 , \ldots , \mathrm{A}_n)) , \qquad (\mathrm{P} \in \mathbb{C}[n\mathrm{End(V)}]) .$$

4.2 $\mathrm{G} = \mathrm{Sp}(2n , \mathbb{C})$ or $\mathrm{O}(n , \mathbb{C})$

Let V be a complex vector space endowed with a non-degenerate symmetric bilinear form B. The map

$$\tau : \mathrm{V} \ni v \to \mathrm{B}(v , \cdot) \in \mathrm{V}^* \tag{2}$$

is an isomorphism of $\mathrm{O(V , B)}$-modules. Obviously, the form $\mathrm{B} \in (\mathrm{V} \otimes \mathrm{V})^*$ is $\mathrm{O(V , B)}$-invariant. We denote by θ the corresponding element in $\mathrm{V} \otimes \mathrm{V}$ coming from the isomorphism given in equation (2).

Let $k = 2j \in \mathbb{N}$ be an even integer and let $\mathrm{V}_k = \mathrm{V}^{\otimes k} = (\mathrm{V} \otimes \mathrm{V})^{\otimes j}$. Let $\theta_k = \otimes^j \theta$. Formally, we have:

$$\theta_k(v_1 \otimes \ldots \otimes v_{2s}) = \prod_{k=1}^{j} \mathrm{B}(v_{2s-1}, v_{2s}).$$

Let's denote by $\lambda_\sigma, \sigma \in \mathscr{S}_k$, the following form

$$\lambda_\sigma(v_1 \otimes v_2 \otimes \ldots \otimes v_k) = \prod_{s=1}^{j} \mathrm{B}(v_{\sigma(2s-1)}, v_{\sigma(2s)}).$$

Obviously, λ_σ is $\mathrm{O}(\mathrm{V}, \mathrm{B})$-invariant.

Theorem 5 – H. Weyl [5]

Any multilinear orthogonal invariant of k-vectors (when k is even) is a linear combinaison of the forms $\lambda_\sigma, \sigma \in \mathscr{S}_k$.

Remark 4.3. The map

$$\mathrm{V} \ni v \longrightarrow \mathrm{B}(v, \cdot) \in V^*$$

is an isomorphism of $\mathrm{O}(\mathrm{V}, \mathrm{B})$-module which induce an isomorphism of $\mathrm{O}(\mathrm{V}, \mathrm{B})$-modules

$$\mathrm{V} \otimes \mathrm{V} \ni v_1 \otimes v_2 \longrightarrow v_1 \otimes \mathrm{B}(v_2, \cdot) \in \mathrm{V} \otimes \mathrm{V}^*.$$

We denote by $\Gamma : \mathrm{V} \otimes \mathrm{V} \to \mathrm{End}(\mathrm{V})$ the corresponding isomorphism of $\mathrm{O}(\mathrm{V}, \mathrm{B})$-module.

Lemma 1

1. For all $u, v \in \mathrm{V}$, we have $\mathrm{tr}(\Gamma(u \otimes v)) = \mathrm{B}(u, v)$.
2. For all $u, v \in \mathrm{V}$, we have: $\Gamma(u \otimes v)^t = \Gamma(v \otimes u)$.

Proof. 1. We fix a basis $\mathscr{B}_\mathrm{V} = \{e_1, \ldots, e_n\}$ of V. Then u can be written as $u = \sum_{k=1}^{n} \alpha_k e_k$ for some scalars $\alpha_k \in \mathbb{C}$. For all $i \in \{1, \ldots, n\}$, we get

$$\Gamma(u \otimes v)(e_i) = \mathrm{B}(v, e_i)u = \sum_{k=1}^{n} \alpha_k \mathrm{B}(v, e_i) v_k.$$

Then

$$\mathrm{tr}(\Gamma(v_1 \otimes v_2)) = \sum_{i=1}^{n} \mathrm{pr}_{\mathbb{C}e_i} \Gamma(v_1 \otimes v_2)(e_i) = \mathrm{B}(v_2, e_i) = \mathrm{B}(v_2, \sum_{i=1}^{n} \alpha_i e_i) = \mathrm{B}(v_2, v_1) = \mathrm{B}(v_1, v_2).$$

2. Fix $u, v \in \mathrm{V}$. So $u = \sum_{i=1}^{n} u_i e_i$, $u = \sum_{i=1}^{n} v_i e_i$. Then

$$\Gamma(u \otimes v)_{i,j} = \mathrm{pr}_{\mathbb{C}e_j}(\Gamma(u \otimes v)(e_i)) = \mathrm{pr}_{\mathbb{C}e_j}(\mathrm{B}(u, e_i)(v)) = \mathrm{B}(u, e_i) v_j = u_i v_j$$

and

$$\Gamma(v \otimes u)_{j,i} = \mathrm{pr}_{\mathbb{C}e_i}(\Gamma(b \otimes u)(e_j)) = \mathrm{pr}_{\mathbb{C}e_i}(\mathrm{B}(v, e_j)(u)) = \mathrm{B}(v, e_j)i_i = u_i v_j \,.$$

□

We denote by $\sigma = (i_1, \ldots, i_r) \in \mathscr{S}_{2k}$ a cycle of length r. We denote by $\mathrm{Supp}(\sigma)$ and $\mathrm{Fix}(\sigma)$ the disjoint subsets of $\{1, \ldots, 2k\}$ given by:

$$\mathrm{Supp}(\sigma) = \{x \in \{1, \ldots, 2k\}, \sigma(x) \neq x\} \qquad \mathrm{Fix}(\sigma) = \{x \in \{1, \ldots, 2k\}, \sigma(x) = x\} \,.$$

For every i_j, $1 \leq j \leq r$, we denote by $[i_j]$ the integer given by:

$$[i_j] = \begin{cases} \mathrm{E}\left(\frac{i_j}{2}\right) + 1 & \text{if } i_j \text{ is odd} \\ \frac{i_j}{2} & \text{otherwise} \end{cases}$$

Proposition 3

For all $\mathrm{A}_1 \otimes \ldots \otimes \mathrm{A}_k \in \mathrm{End(V)}^{\otimes k}$, *we get:*

$$\mu_\sigma(\mathrm{A}_1 \otimes \mathrm{A}_k) = \mathrm{tr}(\prod_{k \in \Omega_\sigma} \mathrm{U}_k) \prod_{n \in \Lambda_\sigma} \mathrm{tr}(\mathrm{A}_n) \,,$$

where

$$\Omega_\sigma = \left\{[i_j], i_j \in \mathrm{Supp}(\sigma)\right\}, \qquad \Lambda_\sigma = \left\{i_j \in 2\mathbb{Z} + 1, \{i_j, i_j + 1\} \in \mathrm{Fix}(\sigma)\right\} \,,$$

and $\mathrm{U}_k = \mathrm{A}_k$ *or* A_k^t.

Proof. Obviously, if there exists an odd element $n \in \{1, \ldots, 2k\}$ such that $\{n, n+1\} \in \mathrm{Fix}(\sigma)$, then $\mathrm{tr}(\mathrm{A}_{[n]})$ will appear in the decomposition of μ_σ.
Without loss of generality, we can assume that our σ does not contain such n. To simplify the notations. we will assume that the sequence i_j, $1 \leq j \leq r$ satisfies $i_1 < i_2 < \ldots < i_r$.
For all $1 \leq a \leq k$, we will denote by w_k either u_k or v_k.

1. If i_1 is odd, we distinguish two different cases:
   (a) If $i_2 \neq i_1 + 1$, then $\mathrm{B}(w_{[i_r]}, v_{[i_1]})$ appears in the decomposition,
   (b) If $i_2 = i_1 + 1$, then $\mathrm{B}(w_{[i_r]}, u_{[i_1]})$ appears in the decomposition,
2. If i_1 is even, then $\mathrm{B}(u_{[i_1]}, w_{[i_r]})$ appears in the decomposition.

1. If i_2 is odd, we distinguish two different cases:
   (a) If $i_3 \neq i_2 + 1$, we distinguish two cases:
      i. If $i_2 = i_1+1$, then i_1 is even and $\mathrm{B}(u_{[i_1]}, w_{[i_r]})\mathrm{B}(v_{[i_1]}, v_{[i_2]})$ appears in the decomposition,
      ii. If $i_2 \neq i_1 + 1$, then i_1 is even and $\mathrm{B}(w_{[i_r]}, v_{[i_1]})\mathrm{B}(u_{[i_2]}, u_{[i_1]})$ appears in the decomposition.

(b) If $i_3 = i_2 + 1$, we distinguish two cases:

i. If $i_2 = i_1 + 1$, then i_1 is even and $\mathrm{B}(u_{[i_1]}, w_{[i_r]})\mathrm{B}(v_{[i_1]}, u_{[i_2]})$ appears in the decomposition,

ii. If $i_2 \neq i_1 + 1$, then i_1 is even and $\mathrm{B}(w_{[i_r]}, v_{[i_1]})\mathrm{B}(u_{[i_1]}, u_{[i_2]})$ appears in the decomposition.

2. If i_2 is even, we distinguish two different cases:

(a) If $i_2 = i_1 + 1$, we distinguish two cases:

i. If $i_3 \neq i_2 + 1$, then $\mathrm{B}(w_{[i_r]}, u_{[i_1]})\mathrm{B}(v_{[i_1]}, v_{[i_2]})$ appears in the decomposition,

ii. If $i_3 = i_1 + 1$, then $\mathrm{B}(w_{[i_r]}, u_{[i_1]})\mathrm{B}(v_{[i_1]}, u_{[i_2]})$ appears in the decomposition.

(b) If $i_2 \neq i_1 + 1$, we distinguish two cases:

i. If i_1 is odd, then $\mathrm{B}(w_{[i_r]}, v_{[i_1]})\mathrm{B}(u_{[i_2]}, u_{[i_1]})$ appears in the decomposition,

ii. If i_1 is even, then $\mathrm{B}(u_{[i_1]}, w_{[i_r]})\mathrm{B}(u_{[i_2]}, v_{[i_1]})$ appears in the decomposition.

Then by continuing the process, we get:

$$\mu_\sigma(\mathrm{A}_1 \otimes \ldots \otimes \mathrm{A}_k) = \Big(\mathrm{B}(w^*_{i_1}, w_{i_2})\mathrm{B}(w^*_{i_2}, w_{i_3}) \ldots \mathrm{B}(w^*_{i_r}, w_{i_1})\Big) \prod_{n \in \Lambda_\sigma} \mathrm{B}(u_{[n]}, v_{[n]}),$$

where

$$w^*_{[i_a]} = \begin{cases} v_{[i_a]} & \text{if } w_{[i_a]} = u_{[i_a]} \\ u_{[i_a]} & \text{if } w_{[i_a]} = v_{[i_a]} \end{cases}.$$

In particular, we get:

$$\mu_\sigma(\mathrm{A}_1 \otimes \ldots \otimes \mathrm{A}_k) = \mathrm{tr}(\prod_{k \in \Omega_\sigma} \mathrm{U}_k) \prod_{n \in \Lambda_\sigma} \mathrm{tr}(\mathrm{A}_n).$$

□

Corollary 4

Let $\sigma = \sigma_1 \ldots \sigma_r$ be a product of permutations such that $\mathrm{Supp}(\sigma_i) \cap \mathrm{Supp}(\sigma_j) =$ if $i \neq j$ (the cardinal of $\mathrm{Fix}(\sigma_i)$ can be one). Then for every $\mathrm{A}_1 \otimes \ldots \otimes \mathrm{A}_k \in \mathrm{End(V)}^{\otimes k}$, we get:

$$\mu_\sigma(\mathrm{A}_1 \otimes \ldots \otimes \mathrm{A}_k) = \prod_{i=1}^{r} \mathrm{tr}\left(\prod_{k \in \Omega_{\sigma_i}} \mathrm{U}_k\right).$$

Theorem 6 – Procesi

The algebra $\mathbb{C}[n\mathrm{End(V)}]^{\mathrm{O(V,B)}}$ *is generated by elements of the form*

$$(\mathrm{A}_1, \ldots, \mathrm{A}_n) \to \mathrm{tr}(\mathrm{P}(\mathrm{A}_1, \ldots, \mathrm{A}_n, \mathrm{A}^t_1, \ldots, \mathrm{A}^t_n)), \qquad (\mathrm{P} \in \mathbb{C}[2n\mathrm{End(V)}]).$$

Finally, let V be a finite dimensional vector space over $\mathbb{C}$ endowed with a non-degenerate, skew-symmetric form B and let Sp(V, B) be the corresponding group of isometries.

For every $A \in End(V)$, there exists a unique element A^* of $End(V)$ such that

$$B(A(u), v) = B(u, A^*(v)), \qquad (u, v \in V).$$

Remark 4.4. We can give an explicit form of A^*. Let $\{e_1, \ldots, e_k, f_1, \ldots, f_k\}$ be a symplectic basis of V, i.e.

$$B(e_i, e_j) = B(f_i, f_j) = 0, \quad B(e_i, f_j) = \delta_{i,j}, \quad (i, j \in [|1, n|]).$$

In particular, we identify $Sp(V, B)$ with the subgroup of $GL(2n, \mathbb{C})$ given by

$$\{g \in GL(2n, \mathbb{C}), g^t J g = J\},$$

where $J = \begin{pmatrix} 0 & Id_n \\ -Id_n & 0 \end{pmatrix}$. In particular, for every $A \in End(V)$, $A^* = Jg^tJ$.

By using the techniques used in the orthogonal case, we get the following theorem.

Theorem 7 – Procesi [2]

The algebra $\mathbb{C}[n End(V)]^{Sp(V,B)}$ *is generated by elements of the form*

$$(A_1, \ldots, A_n) \to tr(P(A_1, \ldots, A_n, A_1^*, \ldots, A_n^*)), \qquad (P \in \mathbb{C}[2n End(V)]).$$

4.3 $G = SO(2n, \mathbb{C})$

We denote by Pf the Pfaffian on $Mat(2n, \mathbb{C})$ (the Pfaffian is usually defined on skew-symmetric matrices but we define Pf on $Mat(2n, \mathbb{C})$ as follow

$$Pf(X) = Pf\left(\frac{X + X^t}{2} + \frac{X - X^t}{2}\right) = Pf\left(\frac{X - X^t}{2}\right), \qquad (X \in Mat(2n, \mathbb{C})).$$

It is know that the Pfaffian Pf satisfies the condition

$$Pf(gXg^t) = \det(g) Pf(X), \qquad (g \in GL(2n, \mathbb{C}), X \in Mat(2n, \mathbb{C})).$$

In particular, for every $g \in SO(2n, \mathbb{C})$, we have

$$Pf(gXg^{-1}) = Pf(X), \qquad (X \in Mat(2n, \mathbb{C})),$$

i.e. Pf if $SO(2n, \mathbb{C})$-invariant.

We denote by $\widetilde{Pf}$ the polarization of the Pfaffian. It is a n-multilinear map satisfying $\widetilde{Pf}(\underbrace{A, \ldots, A}_{n-\text{times}}) = n! Pf(A)$.

Theorem 8 – Aslaksen - Tan - Zhu [6]

The algebra $\mathbb{C}[k Mat(2n, \mathbb{C})]^{SO(2n, \mathbb{C})}$ *is generated by elements of the form*

$$(A_1, \ldots, A_k) \to tr(P(A_1, \ldots, A_k, A_1^t, \ldots, A_k^t))$$

and

$$A = (A_1, \ldots, A_k) \to \widetilde{Pf}(P_1(A, A^t), \ldots, P_n(A, A^t)),$$

where $P, P_1, \ldots, P_n$ *are non commutative polynomials.*

Proof. See [6, Theorem 3]. ∎

Remark 4.5. The reason why we are focusing our attention on the even case is that

$$\mathbb{C}[k\mathrm{Mat}(2n+1,\mathbb{C})]^{\mathrm{SO}(2n+1,\mathbb{C})} = \mathbb{C}[k\mathrm{Mat}(2n+1,\mathbb{C})]^{\mathrm{O}(2n+1,\mathbb{C})}.$$

One can easily see that the previous equality holds by noticing that $\mathrm{Id} \in \mathrm{O}(2n+1,\mathbb{C}) \setminus \mathrm{SO}(2n+1,\mathbb{C})$.

5 Character Variety of classical groups

In this section, we assume that G is a Zariski closed subgroup of SL(V), where V is a finite dimensional vector space over $\mathbb{C}$ and let Γ be a closed subgroup of F_r, with $r \in \mathbb{Z}_+$. Using that G is a Zariski closed subgroup of SL(V), we get that the restriction map

$$\mathbb{C}[\mathrm{End}(\mathrm{V})] \to \mathbb{C}[\mathrm{G}]$$

is surjective. Then the corresponding map:

$$\mathbb{C}[\mathrm{End}(\mathrm{V})]^{\otimes r} \to \mathbb{C}[\mathrm{G}]^{\otimes r}$$

is surjective. Using that $\mathbb{C}[\mathrm{End}(\mathrm{V})]^{\otimes r} \cong \mathbb{C}[\mathrm{End}(\mathrm{V})^{\otimes r}]$, we get a natural surjection:

$$\mathbb{C}[\mathrm{End}(\mathrm{V})^r] \to \mathbb{C}[\mathrm{Hom}(F_r,\mathrm{G})],$$

and because the algebraic variety $\mathrm{Hom}(\Gamma,\mathrm{G})$ is a closed subset of $\mathrm{Hom}(F_r,\mathrm{G}) = \mathrm{G}^r$, we got by composition a natural surjection:

$$\mathbb{C}[\mathrm{End}(\mathrm{V})^r] \to \mathbb{C}[\mathrm{Hom}(\Gamma,\mathrm{G})]. \tag{3}$$

This map corresponding map

$$\Psi : \mathbb{C}[\mathrm{End}(\mathrm{V})^r]^{\mathrm{G}} \to \mathbb{C}[\mathrm{Hom}(\Gamma,\mathrm{G})]^{\mathrm{G}}$$

is well-defined, G-equivariant and surjective. In particular,

$$\Psi(\mathbb{C}[\mathrm{End}(\mathrm{V})^r]^{\mathrm{G}}) = \mathbb{C}[\mathrm{Hom}(\Gamma,\mathrm{G})]^{\mathrm{G}}.$$

Theorem 9

Let $\mathrm{G} = \mathrm{SL}(\mathrm{V}), \mathrm{Sp}(\mathrm{V})$ *or* $\mathrm{G} = \mathrm{SO}(\mathrm{V})$ *with* $\dim(\mathrm{V}) \in 2\mathbb{Z}+1$. *Then*

$$\mathbb{C}[\mathscr{X}(\Gamma,\mathrm{G})] = \mathscr{F}\mathscr{T}(\Gamma,\mathrm{G}) = \mathscr{T}(\Gamma,\mathrm{G}).$$

Proof. As explained before, we have that following surjective maps

$$\mathbb{C}[r\mathrm{End}(\mathrm{V})]^{\mathrm{G}} \to \mathbb{C}[\mathrm{G}^r]^{\mathrm{G}} \to \mathbb{C}[\mathrm{Hom}(\Gamma,\mathrm{G})].$$

The result is obvious for $G = SL(V)$. For $G = Sp(V)$, we know that the elements of the form

$$(A_1, \ldots, A_n) \to \mathrm{tr}\,(P(A_1, \ldots, A_n, A_1^*, \ldots, A_n^*)) , \qquad (P \in \mathbb{C}[2n\mathrm{End}(V)])$$

generate $\mathbb{C}[r\mathrm{End}(V)]^{Sp(V)}$. One can easily see that $A^* = A^{-1}$ if $A \in Sp(V)$. It implies that the elements

$$(A_1, \ldots, A_n) \to \mathrm{tr}\left(P(A_1, \ldots, A_n, A_1^{-1}, \ldots, A_n^{-1})\right) , \qquad (P \in \mathbb{C}[2n\mathrm{End}(V)])$$

are generators of $\mathbb{C}[Sp(V)^r]^{Sp(V)}$ and in particular, we get

$$\mathbb{C}[\mathscr{X}(\Gamma, Sp(V))] = \mathscr{T}(\Gamma, Sp(V)) .$$

The proof is similar for $G = SO(V)$.

□

Remark 5.1. We explain here the link between $\mathscr{X}(F_r, SL(V))$ and $\mathscr{X}(F_r, GL(V))$. We have the following exact sequence:

$$1 \to SL(V) \to GL(V) \to^{\det} \mathbb{C}^* \to 1 .$$

It induces a natural map $\widetilde{\det} : \mathrm{Hom}(F_r, GL(V)) \to \mathrm{Hom}(F_r, \mathbb{C}^*) \approx \mathbb{C}^{*r}$ given by

$$\widetilde{\det}(\rho) = (\det(\rho(X_1)), \ldots, \det(\rho(X_r))) ,$$

where $X_1, \ldots, X_r$ are the generators of the free group. The map induced on $\mathscr{X}(F_r, GL(V))$ is well-defined, and because $\mathbb{C}^*$ is abelian, we get that $\mathbb{C}^{*r} \approx \mathscr{X}(F_r, \mathbb{C}^*)$.
We get that $\mathscr{X}(F_r, SL(V)) = \widetilde{\det}^{-1}(1_r)$, where $1_r = (1, \ldots, 1)$ and we get the following exact sequence:

$$1 \to \mathscr{X}(F_r, SL(V)) \to \mathscr{X}(F_r, GL(V)) \to \mathscr{X}(F_r, \mathbb{C}^*) \to 1 .$$

Theorem 10 – (A. Sikora)

Let $G = SO(V)$, *with* $\dim(V) \in 2\mathbb{Z}$. *Then*

$$\mathscr{F}\mathscr{T}(F_2, G) \not\subset \mathbb{C}[\mathrm{Hom}(F_2, G)] .$$

Proof. See [3, Theorem 1].

□

6 The case $\Gamma = F_2$ and $G = \mathrm{Spin}(4, \mathbb{C})$

As mentionned in the previous section, we have $\mathscr{F}\mathscr{T}(F_2, SO(4, \mathbb{C})) \not\subset \mathbb{C}[\mathrm{Hom}(F_2, SO(4, \mathbb{C}))]$. In this section, we are going to prove that

$$\mathscr{F}\mathscr{T}(F_2, \mathrm{Spin}(4, \mathbb{C})) = \mathbb{C}[\mathrm{Hom}(F_2, \mathrm{Spin}(4, \mathbb{C}))] .$$

We denote by $\widetilde{G}$ the cartesian product $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$. The group $\widetilde{G}$ acts on $V = Mat(2,\mathbb{C})$ naturally:

$$\phi((g,h))(A) = gAh^{-1} \qquad \left((g,h) \in \widetilde{G}, A \in V\right) .$$

On V, we define the quadratic form Q given by $Q(X) = 2\det(X)$ and by polarisation, we get the form B given by $B(X,Y) = \frac{1}{2}(Q(X+Y) - Q(X) - Q(Y))$. Then for every $(g,h) \in \widetilde{G}$ and $X, Y \in V$, we have:

$$\begin{aligned} & B(\phi((g,h))X, \phi(g,h)Y) = \beta(gXh^{-1}, gYh^{-1}) \\ = & \frac{1}{2}\left(\det(gXh^{-1} + gYh^{-1}) - \det(gXh^{-1}) - \det(gYh^{-1})\right) \\ = & \left(\det(g)\det(X+Y)\det(h^{-1}) - \det(g)\det(X)\det(h^{-1}) - \det(g)\det(Y)\det(h^{-1})\right) \\ = & \det(X+Y) - \det(X) - \det(Y) = B(X,Y), \end{aligned}$$

i.e. $\phi((g,h)) \in G(V,B)$. The form B is symmetric and non-degenerate. Let $\mathscr{B}$ be the basis of V given by

$$\mathscr{B} = \{v_1 = E_{1,1}, v_2 = E_{1,2}, v_3 = -E_{2,1}, v_4 = E_{2,2}\} .$$

One can easily see that

$$Mat_{\mathscr{B}}(B) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} . \tag{4}$$

If we identify V with $\mathbb{C}^4$ by sending the vector v_i on e_i (the i-th element of the canonical basis of $\mathbb{C}^4$), the form B becomes B_1, the natural symmetric form on $\mathbb{C}^4$ whose matrix in $\{e_1, e_2, e_3, e_4\}$ is given by (4). In particular,

$$\phi : \widetilde{G} \to O(V,B)$$

is a well-defined homomorphism. Moreover, one can see that $\det(\phi(\tilde{g})) = 1$ for every $\tilde{g} \in \widetilde{G}$, and then

$$\phi : \widetilde{G} \to SO(V,B) .$$

Let $\widetilde{H} = \left\{diag(h_1, h_1^{-1}), h_1 \in \mathbb{C}^*\right\} \times \left\{diag(h_2, h_2^{-1}), h_2 \in \mathbb{C}^*\right\} \subseteq \widetilde{G}$ and H be the subgroup of G given by $\left\{diag(h_1, h_2, h_2^{-1}, h_1^{-1}), h_1, h_2 \in \mathbb{C}^*\right\}$.

Theorem 11

The map ϕ is surjective and $\phi(\widetilde{H}) = H$. Moreover, $Ker(\phi) = \{(1,1),(-1,-1)\} \cong \mathbb{Z}_2$, *i.e.* $\widetilde{G} \cong Spin(V,B)$.

In particular, we have:

$$\begin{aligned} \mathbb{C}[Hom(F_2, Spin(4,\mathbb{C}))]^{Spin(4,\mathbb{C})} &= \mathbb{C}[Spin(4,\mathbb{C})^2]^{Spin(4,\mathbb{C})} = \mathbb{C}[(SL(2,\mathbb{C}) \times SL(2,\mathbb{C}))^2]^{SL(2,\mathbb{C})\times SL(2,\mathbb{C})} \\ &= \mathbb{C}[SL(2,\mathbb{C}) \times SL(2,\mathbb{C})]^{SL(2,\mathbb{C})} \otimes \mathbb{C}[SL(2,\mathbb{C}) \times SL(2,\mathbb{C})]^{SL(2,\mathbb{C})} . \end{aligned}$$

According to the Theorem of Fricke-Vogt, we get that:

$$\mathbb{C}[SL(2,\mathbb{C}) \times SL(2,\mathbb{C})]^{SL(2,\mathbb{C})} \otimes \mathbb{C}[SL(2,\mathbb{C}) \times SL(2,\mathbb{C})]^{SL(2,\mathbb{C})} = \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3],$$

where

$$x_1(A_1, A_2, B_1, B_2) = \mathrm{tr}(A_1), \qquad x_2(A_1, A_2, B_1, B_2) = \mathrm{tr}(B_1), \qquad x_3(A_1, A_2, B_1, B_2) = \mathrm{tr}(A_1B_1),$$

$$y_1(A_1, A_2, B_1, B_2) = \mathrm{tr}(A_2), \qquad y_2(A_1, A_2, B_1, B_2) = \mathrm{tr}(B_2), \qquad y_3(A_1, A_2, B_1, B_2) = \mathrm{tr}(A_2B_2).$$

We denote by Π the spinorial representation of $\mathrm{Spin}(4, \mathbb{C})$. The representation Π is the direct sum of two irreducible representations $\Pi = \Pi_+ \oplus \Pi_-$. We denote by $\widetilde{x_1}, \widetilde{x_2}, \widetilde{x_3}, \widetilde{y_1}, \widetilde{y_2}, \widetilde{y_3}$ the corresponding set of generators of $\mathbb{C}[\mathrm{Spin}(4, \mathbb{C})^2]^{\mathrm{Spin}(4,\mathbb{C})}$. We have:

$$\widetilde{x_1}(\widetilde{g}, \widetilde{h}) = \mathrm{tr}(\Pi_+(\widetilde{g})), \qquad \widetilde{x_2}(\widetilde{g}, \widetilde{h}) = \mathrm{tr}(\Pi_+(\widetilde{h})), \qquad \widetilde{x_3}(\widetilde{g}, \widetilde{h}) = \mathrm{tr}(\Pi_+(\widetilde{gh})),$$

$$\widetilde{y_1}(\tilde{g}, \tilde{h}) = \mathrm{tr}(\Pi_-(\tilde{g})), \qquad \widetilde{y_2}(\tilde{g}, \tilde{h}) = \mathrm{tr}(\Pi_-(\tilde{h})), \qquad \widetilde{y_3}(\tilde{g}, \tilde{h}) = \mathrm{tr}(\Pi_-(\tilde{g}\tilde{h})).$$

In particular, we have:

$$\mathscr{X}(F_2, \mathrm{Spin}(4, \mathbb{C})) = \mathscr{F}\mathscr{T}(F_2, \mathrm{Spin}(4, \mathbb{C})).$$

Conjecture 1

For every $n \geq 2$, we get:

$$\mathscr{F}\mathscr{T}(F_2, \mathrm{Spin}(2n, \mathbb{C})) = \mathbb{C}[\mathscr{X}(F_2, \mathrm{Spin}(2n, \mathbb{C}))].$$

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