Howe Duality and Characters

Allan Merino (University of Ottawa)
Winter Young Mathematician Forum
Shanghai Jiao Tong University

December 9, 2022

G: Lie group

- A representation of G is a pair $(\Pi, \mathscr{H})$ such that
  - $\mathscr{H}$ is a (complex) Hilbert space ,
  - $\Pi : \mathrm{G} \to \mathrm{GL}(\mathscr{H})$ is a group morphism, where $\mathrm{GL}(\mathscr{H})$ is the group of invertible operators on $\mathscr{H}$,
  - The map

$$\mathrm{G} \times \mathscr{H} \ni (g, v) \to \Pi(g)(v) \in \mathscr{H}$$

    is continuous.
- The representation is called:
  - $(\Pi, \mathscr{H})$ irreducible: no closed G-invariant subspaces in $\mathscr{H}$
  - $(\Pi, \mathscr{H})$ unitary: $\langle \Pi(g)u, \Pi(g)v \rangle = \langle u, v \rangle$, $g \in \mathrm{G}$, $u, v \in \mathscr{H}$.

Main Question

Find/Parametrize (up to equivalence) the set $\widehat{\mathrm{G}}$ of irreducible unitary representations of G

In some cases, we have an explicit parametrization of $\widehat{G}$:

1. (H. Weyl) - G is a compact Lie group
   e.g. $G = O(n, \mathbb{R}), U(n, \mathbb{C}), U(n, \mathbb{H}), \cdots$

$$\widehat{G} \leftrightarrow \text{ lattice in } i\mathfrak{t}^*$$

2. (A. Kirillov) - G is a simply connected, connected, nilpotent Lie group
   e.g. H_n (Heisenberg group)

$$\widehat{G} \leftrightarrow \mathfrak{g}^*/G = \text{ co-adjoint orbits}$$

3. (D. Barbasch) - G classical complex groups
   e.g. $G = GL(n, \mathbb{C}), SL(n, \mathbb{C}), Sp(2n, \mathbb{C}), (S)O(n, \mathbb{C})$

Most interesting case

G: real reductive Lie group
e.g. $G = GL(n, \mathbb{R}), Sp(2n, \mathbb{R}), (S)O(p, q, \mathbb{R}), U(p, q, \mathbb{C}), G_2, E_8, \ldots$

G: real reductive Lie group, K: maximal compact subgroup of G,

- $(\mathrm{G}, \mathrm{K}) = (\mathrm{GL}(n, \mathbb{R}), \mathrm{O}(n, \mathbb{R}))$
- $(\mathrm{G}, \mathrm{K}) = (\mathrm{O}(p, q, \mathbb{R}), \mathrm{O}(p, \mathbb{R}) \times \mathrm{O}(q, \mathbb{R}))$

Let $(\Pi, \mathscr{H})$ be an irreducible representation of G. Then

$$\Pi_{|_{\mathrm{K}}} = \bigoplus_{(\lambda, \mathrm{V}_\lambda) \in \widehat{\mathrm{K}}} m_\lambda \lambda .$$

Admissible Representation

The representation Π is admissible if $m_\lambda < \infty$ for every $\lambda \in \widehat{\mathrm{K}}$.

Notation: $\widehat{\mathrm{G}}_{\mathrm{adm}}$ the set of (equivalence classes) of irreducible admissible representations of G.

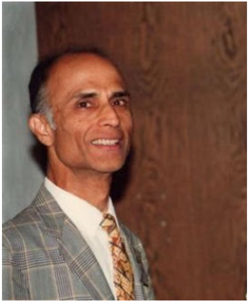
Theorem (Harish-Chandra)

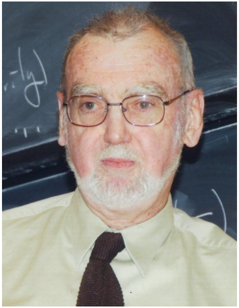
Every irreducible unitary representation of G is admissible.

$$\widehat{\mathrm{G}} \subseteq \widehat{\mathrm{G}}_{\mathrm{adm}}$$

Langlands, '73

Parametrization of the irreducible admissible representations of G.





We have many different techniques:

1. Restriction

$$\mathrm{H} \subseteq \mathrm{G}\,, (\Pi\,, \mathscr{H}) \text{ irred. rep. of } \mathrm{G} \rightsquigarrow \Pi_{|\mathrm{H}} \text{ rep. of } \mathrm{H}$$

2. Induction

$$\mathrm{H} \subseteq \mathrm{G}\,, (\sigma\,, \mathrm{V}) \text{ irred. rep. of } \mathrm{H} \rightsquigarrow \mathrm{Ind}_{\mathrm{H}}^{\mathrm{G}}(\sigma) \text{ rep. of } \mathrm{G}$$

3. Duality
    1. Schur-Weyl duality ($\mathscr{S}_d \leftrightarrow \mathrm{GL}(n\,, \mathbb{C})$) ,
    2. Howe duality ($\mathrm{G} \leftrightarrow \mathrm{G}'$, where $(\mathrm{G}\,, \mathrm{G}')$ is a dual pair in a symplectic group)
    3. " Langlands program "

Characters of admissible representations

$(\Pi, \mathscr{H})$: representation of G.

- If $\dim(\mathscr{H}) < \infty$,

$$\Theta_\Pi : \mathrm{G} \ni g \to \mathrm{tr}(\Pi(g)) \in \mathbb{C}$$

is well-defined.

- General case (Π admissible , Harish-Chandra, 1954 - 1963): For every $\Psi \in \mathscr{C}_c^\infty(\mathrm{G})$, the operator

$$\Pi(\Psi) = \int_{\mathrm{G}} \Psi(g)\Pi(g)dg$$

is a trace class operator and the map

$$\Theta_\Pi : \mathscr{C}_c^\infty(\mathrm{G}) \ni \Psi \to \mathrm{tr}(\Pi(\Psi)) \in \mathbb{C}$$

is a distribution (in the sense of Laurent Schwartz).

Moreover, there exists a locally integrable function Θ_Π on G, analytic on $\mathrm{G}^{\mathrm{reg}}$, such that

$$\Theta_\Pi(\Psi) = \int_{\mathrm{G}} \Psi(g)\Theta_\Pi(g)dg \qquad (\Psi \in \mathscr{C}_c^\infty(\mathrm{G}))\,.$$

In few cases, the function Θ_Π is well-known

- G compact (H. Weyl) ,
- $(\Pi, \mathscr{H})$ discrete series representation (Harish-Chandra , Schmidt-Hecht , Rossmann) ,
- $(\Pi, \mathscr{H})$ irreducible unitary highest weight module (Enright, Merino) .

Howe Duality

$(\mathrm{W}, \langle \cdot\,, \cdot \rangle)$ symplectic space (over $\mathbb{R}$) ,
$\mathrm{Sp}(\mathrm{W}) = \{g \in \mathrm{GL}(\mathrm{W})\,;\langle g(w_1)\,, g(w_2)\rangle = \langle w_1\,, w_2\rangle\,, w_1\,, w_2 \in \mathrm{W}\}$,
$\widetilde{\mathrm{Sp}}(\mathrm{W})$: (connected) double cover of $\mathrm{Sp}(\mathrm{W})$,
$(\omega, \mathscr{H})$: Weil's representation (or metaplectic representation)

Dual pairs

We say that $(\mathrm{G}, \mathrm{G}')$ is a dual pair in $\mathrm{Sp}(\mathrm{W})$ if $\mathscr{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{G}) = \mathrm{G}'$ and $\mathscr{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{G}') = \mathrm{G}$.

Example:

For every subgroup H of $\mathrm{Sp}(\mathrm{W})$,

$$\left(\mathscr{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{H})\,, \mathscr{C}_{\mathrm{Sp}(\mathrm{W})}(\mathscr{C}_{\mathrm{Sp}(\mathrm{W})}(\mathrm{H}))\right)$$

is a dual pair in $\mathrm{Sp}(\mathrm{W})$.

- (G, G') reductive if the actions $G, G' \curvearrowright W$ are semisimple.
- (G, G') irreducible if we cannot find an orthogonal decomposition $W = W_1 \oplus W_2$ with $G \cdot G'$-invariant W_1, W_2.

Example:

$\mathbb{D} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, ι: involution on $\mathbb{D}$

- V: left vector space over $\mathbb{D}$,
- γ: $(\iota, 1)$-hermitian form on V,
- W: right vector space over $\mathbb{D}$,
- γ': $(\iota, -1)$-hermitian form on W,

$$(U(V, \gamma), U(W, \gamma')) \subseteq Sp((W \otimes_{\mathbb{D}} V)_{\mathbb{R}}, Re(\gamma' \otimes \gamma))$$

$$(GL(V), GL(W)) \subseteq GL(W \otimes_{\mathbb{D}} V) \subseteq Sp(W \otimes_{\mathbb{D}} V \oplus (W \otimes_{\mathbb{D}} V)^*).$$

Theorem (Howe)

1. $(\mathrm{O}(p,q,\mathbb{R}),\mathrm{Sp}(2n,\mathbb{R})) \subseteq \mathrm{Sp}(2n(p+q),\mathbb{R})$,
2. $(\mathrm{U}(p,q,\mathbb{C}),\ \mathrm{U}(r,s,\mathbb{C})) \subseteq \mathrm{Sp}(2(p+q)(r+s),\mathbb{R})$,
3. $(\mathrm{O}(m,\mathbb{C}),\mathrm{Sp}(2n,\mathbb{C})) \subseteq \mathrm{Sp}(4nm,\mathbb{R})$,
4. $(\mathrm{Sp}(p,q,\mathbb{H}),\mathrm{O}^*(2n,\mathbb{H})) \subseteq \mathrm{Sp}(4n(p+q),\mathbb{R})$,
5. $(\mathrm{GL}(n,\mathbb{R}),\mathrm{GL}(m,\mathbb{R})) \subseteq \mathrm{Sp}(2nm,\mathbb{R})$,
6. $(\mathrm{GL}(n,\mathbb{C}),\mathrm{GL}(m,\mathbb{C})) \subseteq \mathrm{Sp}(4nm,\mathbb{R})$,
7. $(\mathrm{GL}(n,\mathbb{H}),\mathrm{GL}(m,\mathbb{H})) \subseteq \mathrm{Sp}(8nm,\mathbb{R})$.

Howe duality - Compact pairs

- $(\mathrm{G}, \mathrm{G}')$ irred. reductive dual pair - G compact (e.g. $(\mathrm{U}(n), \mathrm{U}(p, q))$)
- $(\omega, \mathscr{H})$: Weil representation, $\mathscr{H}^\infty$: smooth vectors

The action of $\mathrm{G} \curvearrowright \mathscr{H}^\infty$ is semisimple!

$$\mathscr{H}^\infty = \bigoplus_{(\Pi, \mathrm{V}_\Pi) \in \widetilde{\mathrm{G}}_{\mathrm{irr}}} \mathrm{V}(\Pi), \qquad \underbrace{\mathrm{V}(\Pi) = \mathrm{V}_\Pi \oplus \mathrm{V}_\Pi \oplus \ldots}_{\Pi-\text{ isotypic component}}$$

$$\widetilde{\mathrm{G}} \curvearrowright \mathrm{V}(\Pi) \curvearrowleft \widetilde{\mathrm{G}}'$$

Theorem (Kashiwara/Vergne - Howe)

As a $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}}'$-module, we get

$$\mathrm{V}(\Pi) = \Pi \otimes \Pi'$$

where Π' is an irreducible unitary representation of $\widetilde{\mathrm{G}}'$.

Howe Duality

We have a one-to-one correspondence

$$\widetilde{\mathrm{G}}_{\mathrm{irr}} \ni \Pi \to \Pi' = \theta(\Pi) \in \widetilde{\mathrm{G}}'_{\mathrm{irr}}$$

The correspondence $\Pi \leftrightarrow \Pi'$ is well-understood: both Π and Π' are irreducible unitary highest weight representations!

Theorem (Enright-Howe-Wallach)

Let Π' be an irreducible unitary highest weight representation of $\widetilde{\mathrm{U}}(p, q)$. There exists $n \in \mathbb{Z}_+$ and an irreducible unitary representation Π of $\widetilde{\mathrm{U}}(n)$ such that $\Pi' = \theta(\Pi)$.

Howe Duality - General Case

$(\mathrm{G}, \mathrm{G}')$ irreducible reductive dual pair in $\mathrm{Sp}(\mathrm{W})$

$\mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ = set of conjugacy classes of irreducible admissible representations $(\Pi, \mathscr{H}_\Pi)$ of $\widetilde{\mathrm{G}}$ which can be realized as a quotient of $\mathscr{H}^\infty$ by a closed $\omega^\infty(\widetilde{\mathrm{G}})$-invariant subspace

$\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$, $\Pi \cong \mathscr{H}^\infty / \mathscr{N}$

$$\mathscr{H}(\Pi) = \mathscr{H}^\infty / \left(\bigcap_{\Pi \cong \mathscr{H}^\infty / \mathscr{N}} \mathscr{N} \right)$$

We get that $\widetilde{\mathrm{G}} \curvearrowright \mathscr{H}(\Pi) \curvearrowleft \widetilde{\mathrm{G}'}$ and then

$$\mathscr{H}(\Pi) = \Pi \otimes \Pi'_1$$

⚠ Π'_1 is not irreducible (as a $\widetilde{\mathrm{G}'}$-module) in general!

Theorem (Howe)

The representation Π'_1 has a unique irreducible quotient Π' so that $\Pi' \in \mathscr{R}(\widetilde{\mathrm{G}}', \omega)$.

Theorem (Howe)

We have a one-to-one correspondence

$$\theta : \mathscr{R}(\widetilde{\mathrm{G}}, \omega) \ni \Pi \to \Pi' \in \mathscr{R}(\widetilde{\mathrm{G}'}, \omega) .$$

Philosophy behind the map θ

Starting from a representation Π of $\widetilde{\mathrm{G}}$, we try to understand the corresponding representation of Π' of $\widetilde{\mathrm{G}}'$

" Transfer of the information we have on Π through θ "

Howe duality in Representation Theory

1. (J-S Li) Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair in the stable range

$$e.g. (\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p, q), \mathrm{U}(r, s)), \qquad p + q \leq \min(r, s)$$

Assume that Π is unitary. Then the corresponding representation $\Pi' = \theta(\Pi)$ is unitary!

2. (H-Y Loke - J-J Ma) Let $(\mathrm{G}, \mathrm{G}')$ be a dual pair in the stable range. Then $\Pi' = \Pi'_1$.

3. Let Π' be an irreducible (admissible) representation of $\widetilde{\mathrm{U}}(r, s)$. There exists $p, q \in \mathbb{Z}_+$ such that $\Pi' = \theta(\Pi)$, where Π is an irreducible (admissible) representation of $\widetilde{\mathrm{U}}(p, q)$.

Correspondence of Characters

Question

Can we understand the transfer $\Pi \to \Pi'$ by looking at the correspondence of characters $\Theta_\Pi \to \Theta_{\Pi'}$?

$$\begin{array}{ccc} \Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega) & \longrightarrow & \Pi' \in \mathscr{R}(\widetilde{\mathrm{G}'}, \omega) \\ \downarrow & & \downarrow \\ \Theta_\Pi \in \mathscr{D}'(\widetilde{\mathrm{G}}) & \longrightarrow & \Theta_{\Pi'} \in \mathscr{D}'(\widetilde{\mathrm{G}'}) \end{array}$$

$$\begin{array}{ccc} \Theta_\Pi \in \mathscr{D}'(\widetilde{\mathrm{G}}) & \longrightarrow & \Theta_{\Pi'} \in \mathscr{D}'(\widetilde{\mathrm{G}'}) \\ \downarrow & \nearrow & \downarrow \\ \Theta_\Pi \in \mathrm{L}^1_{\mathrm{loc}}(\widetilde{\mathrm{G}}) & \longrightarrow & \Theta_{\Pi'} \in \mathrm{L}^1_{\mathrm{loc}}(\widetilde{\mathrm{G}'}) \end{array}$$

Transfer of Characters - G compact

Allan Merino, *Transfer of characters in the theta correspondence with one compact member*, **J. Lie Theory** 30 (2020), no. 4, 997-1026.

Idea: For every $(\Pi, V_\Pi) \in \widehat{\mathrm{G}}$, and $\Pi' = \theta(\Pi)$, we get:

$$\Theta_{\Pi'}(\widetilde{g}') = \int_{\widetilde{\mathrm{G}}} \overline{\Theta_\Pi(\widetilde{g})}\Theta(\widetilde{g}\widetilde{g}')d\widetilde{g}$$

Allan Merino, *Characters of some unitary highest weight representations via the theta correspondence*, **J. Funct. Anal.** 279 (2020), no. 8, 108698, 70.

- Based on a result of Rossmann - Duflo - Vergne on the Fourier transform of co-adjoint orbits.

Conjecture of T. Przebinda

$(\mathrm{G}, \mathrm{G}')$ reductive dual pair in $\mathrm{Sp}(\mathrm{W})$, $\mathrm{rk}(\mathrm{G}) \leq \mathrm{rk}(\mathrm{G}')$

$$e.g.\ (\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p, q), \mathrm{U}(r, s)), \qquad p + q \leq r + s$$

$\mathrm{H}_1, \dots, \mathrm{H}_n$ maximal set of non-conjugate Cartan subgroups of G
T. Przebinda constructed a family of distributions $\left\{\mathrm{Chc}_{\widetilde{h}_i}, \widetilde{h}_i \in \widetilde{\mathrm{H}}_i^{\mathrm{reg}}\right\}$ of $\widetilde{\mathrm{G}'}$.
For every $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$ and $\Psi \in \mathscr{C}_c^\infty(\widetilde{\mathrm{G}'})$,

$$\mathrm{Chc}^*(\Theta_\Pi)(\Psi) = \sum_{i=1}^n c_i \int_{\widetilde{\mathrm{H}}_i} \overline{\Theta_\Pi(\widetilde{h}_i)} |\mathrm{D}(\widetilde{h}_i)|^2 \mathrm{Chc}_{\widetilde{h}_i}(\Psi) d\widetilde{h}_i$$

is a well-defined distribution on $\widetilde{\mathrm{G}'}$.

Conjecture (Przebinda)

For every $\Pi \in \mathscr{R}(\widetilde{\mathrm{G}}, \omega)$, $\mathrm{Chc}^*(\Theta_\Pi) = \Theta_{\Pi'_1}$.

What is known about Przebinda's conjecture?

- $(\mathrm{G}, \mathrm{G}')$ with G compact,
- (Przebinda) $(\mathrm{G}, \mathrm{G}')$ a dual pair in the stable range.

Allan Merino, *Transfer of characters for discrete series representations of the unitary groups in the equal rank case via the Cauchy-Harish-Chandra integral*, **International Mathematics Research Notices** (IMRN, 2022).

- Proof of the Conjecture for the dual pair $(\mathrm{G}, \mathrm{G}') = (\mathrm{U}(p, q), \mathrm{U}(r, s))$, with $p + q = r + s$, starting from a discrete series representation.

Allan Merino, *Distribution character of the theta lift of a discrete series representations*, **In Preparation**

- Proof of the Conjecture for the dual pair $(\mathrm{G}, \mathrm{G}')$, with $\mathrm{rk}(\mathrm{G}) \leq \mathrm{rk}(\mathrm{G}')$, starting from a discrete series representation.

Concluding Remarks

1. The duality works for p-adic groups as well (Walspurger, Gan-Takeda, Gan-Sun).
2. Why do we consider the Weil representation?

   "Because it works!"

   The Weil representation is minimal.
3. Extension to minimal representation of other semisimple Lie groups (J-S Li, Savin, etc ...)
4. Extension to Lie supergroups (Howe, Nishiyama, Cheng, Wang, Coulembier, Lavicka, Soucek, Lu, Salmasian, Merino)

謝謝